A Bidirectional Decision Procedure for Intuitionistic Modal Logic IS5*

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In this paper, we present a new label-free multi-context sequent calculus for intuitionistic modal logic **IS5** and develop its bidirectional decision procedure. Our sequent calculus uses a new form of label-free sequent which has three different contexts—a local context for describing the current world, a global context containing universally valid formulas, and a frame for describing accessible worlds. Its bidirectional decision procedure first simulates forward proof search in order to generate a set of derived inference rules specialized for the query formula. Then it attempts to build a proof tree composed of these inference rules by performing proof search in a backward direction. Thus, although it internally performs only backward proof search, the decision procedure is effectively bidirectional, thereby taking advantage of both forward and backward proof search. We have formalized our sequent calculus and its cut elimination property in the Coq proof assistant and implemented an automated theorem prover based on our bidirectional decision procedure in OCaml.

CCS Concepts: • **Theory of computation → Proof theory**; **Modal and temporal logics**; **Automated reasoning**;

Additional Key Words and Phrases: Intuitionistic modal logic, sequent calculus, cut elimination, focusing, bidirectional proof search, automated theorem proving

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1 INTRODUCTION

Modal logics extend propositional or predicate logics with modal operators such as necessity $\Box$ or possibility $\Diamond$, allowing us to describe different modes of truth such as "a proposition A is always true" or "A is true at some point". More formally, based on the abstract notion of worlds and some accessibility relation defined on them, $\Box A$ is true if A is true in *every* world accessible from the current one, and $\Diamond A$ is true if A is true in *some* world accessible from the current one.

Among various modal logics, intuitionistic modal logics have inspired a number of applications in the design of programming languages via the Curry-Howard correspondence, that is, by interpreting propositions as types and instantiating the notion of worlds properly. Examples include type systems

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for staged computation [5, 14] and distributed computations [13, 22] and task description languages for computer networks [3], to name a few. Modal logics often used in these contexts are intuitionistic modal logics **IS4** or **IS5**.

Although proof-theoretic properties and semantics of intuitionistic modal logics such as **IS4** and **IS5** have been extensively studied, for example in [2, 7, 8, 18–21], efficient theorem proving and proof search methods for these logics have received a little attention. A notable exception is the work by Heilala and Pientka [10]. They present a label-free multi-context sequent calculus for the propositional fragment of **IS4** and develop its bidirectional decision procedure which first generates a set of derived inference rules relevant to the query formula using forward proof search, and then constructs a proof tree composed of these inference rules in a backward direction. As for **IS5**, Galmiche and Salhi [8] propose a label-free sequent calculus for the propositional fragment of **IS5** but they present only a simple, exhaustive proof search algorithm based on the subformula property of their cut-free sequent calculus.

In this paper, following the approach of Heilala and Pientka [10], we propose an effective and efficient bidirectional decision procedure for the propositional fragment of **IS5**. Specifically, we first design a new cut-free sequent calculus $\mathbf{G_{IS5}}$ based on a new form of label-free sequent which has three different contexts—a local context for describing the current world, a global context containing universally valid formulas, and a frame for describing accessible worlds (Section 2). Our multi-context sequent extends the MC-sequent proposed in [8] with a global context, and the use of the three contexts makes our sequent calculus more suitable for studying and developing a bidirectional proof search method for **IS5**. (The bidirectional method proposed in [10] is also built on a sequent calculus using local and global contexts.) We then propose a refined sequent calculus $\mathbf{G^F_{IS5}}$ using a focusing mechanism and a signed subformula property, which is suitable for forward proof search (Section 3). Finally, we develop a bidirectional decision procedure for **IS5** based on $\mathbf{G^F_{IS5}}$ (Section 4). More precisely, given a query formula in **IS5**, it first simulates proof search in a forward direction based on the focusing mechanism and the signed subformula property, and generates a set of derived inference rules specialized for the query formula. Then it attempts to build a proof tree composed of these inference rules by performing proof search in a backward direction. Thus, although it internally performs only backward proof search, our decision procedure is effectively bidirectional, thereby taking advantage of both forward and backward proof search.

Our contributions are summarized as follows:

- We propose a new label-free multi-context sequent calculus $\mathbf{G_{IS5}}$ for the propositional fragment of **IS5** and formally prove its cut elimination property using the Coq proof assistant. We also prove that $\mathbf{G_{IS5}}$ is sound and complete with respect to the Kripke semantics of **IS5**.
- We propose a refined, focused sequent calculus for forward proof search and develop a sound and complete bidirectional decision procedure for **IS5**.
- Finally, we implemented an automated theorem prover based on our bidirectional decision procedure in OCaml. Both the Coq proof scripts and the OCaml implementation are available at http://pl.postech.ac.kr/IS5/.

2 A LABEL-FREE MULTI-CONTEXT SEQUENT CALCULUS FOR IS5

2.1 Label-Free Multi-Context Sequents

In addition to logical connectives of the intuitionistic propositional logic, formulas in **IS5** also include modal operators $\Box$ and $\Diamond$ as follows:

$$\text{formulas} \quad A, B, C \quad ::= \quad p \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A$$

where p represents atomic propositional constants which cannot be destructed into subformulas. The negation and truth are defined notionally in the usual way: $\neg A \triangleq A \supset \bot$ and $\top \triangleq \neg\bot$. The meaning of the propositional connectives is the same as in the intuitionistic propositional logic. Based on the abstract notion of worlds and some accessibility relation defined on them, which is an equivalence relation in **IS5**, $\Box A$ means that A is true in every world accessible from the current one, and $\Diamond A$ means that A is true in some world accessible from the current one. As usual, $\supset$ is right-associative and has the lowest precedence, and $\Box$ and $\Diamond$ have a higher precedence than $\wedge$ and $\vee$.

Our sequent calculus uses the following form of label-free multi-context sequents:

$$\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$$

where Γ, Δ, Σ are defined as follows:

$$\begin{array}{rcll} \text{local contexts} & \Gamma & ::= & \cdot \mid \Gamma, A \\ \text{global contexts} & \Delta & ::= & \cdot \mid \Delta, A \\ \text{frames} & \Sigma & ::= & \cdot \mid \Sigma; \Gamma \end{array}$$

Contexts and frames are multisets of formulas and contexts, respectively. We use "·" to indicate both an empty context and an empty frame. We use Γ to represent a local context of true hypotheses and Δ to represent a global context of *valid* hypotheses. The local context Γ in a sequent $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ represents true hypotheses in the current world where the provability of the succedent C is to be established. Each context in the frame Σ represents a world accessible from and to the current one, and the contexts in Σ are accessible from each other. Two different contexts in Σ do not necessarily represent two different worlds. Moreover, a formula that is true in some context may not be true in other contexts. The formulas in the global context Δ are assumed to be true in every world accessible from the current one, and said to be valid. A sequent $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ is then simply read "C can be proved using all assumptions Σ, Δ, Γ" and can be interpreted as $(\Diamond(\bigwedge \Gamma_1) \wedge \cdots \wedge \Diamond(\bigwedge \Gamma_n)) \supset \Box(\bigwedge \Delta) \supset (\bigwedge \Gamma) \supset C$ when $\Sigma = \Gamma_1; \ldots; \Gamma_n$. We use $\bigwedge \Gamma$ to denote $A_1 \wedge \cdots \wedge A_n$ when $\Gamma = A_1, \ldots, A_n$. As a final remark, although the frame and the local context are sufficient to express the accessibility relation as an equivalence relation in a sequent calculus for **IS5** [8], the use of global contexts makes it simpler to deal with modality $\Box$ as illustrated in the next subsection.

2.2 A Sequent Calculus $\mathbf{G_{IS5}}$

Figure 1 shows our sequent calculus $\mathbf{G_{IS5}}$ for **IS5**, based on the definition of label-free multi-context sequents. It mainly consists of left and right rules for each connective, which respectively correspond to elimination and introduction rules in natural deduction systems. The name of each inference rule is composed of the connective of the principal formula, followed by a letter L for left rules or R for right rules, with a superscript for left rules indicating the context in which the principal formula resides. The exception is the Id rules which prove an atomic proposition p using a hypothesis p in either the local context or the global context.

The inference rules for the propositional connectives are mostly standard. A left rule analyzes a formula in some context, *eliminating* the top-level connective of the formula, *e.g.*, $\wedge\mathsf{L}$, $\vee\mathsf{L}^{local}$, $\vee\mathsf{L}^{frame}$, $\supset\mathsf{L}^{local}$, $\supset\mathsf{L}^{frame}$, when reading the rule in a bottom up way. In contrast, a right rule *introduces* a connective in the succedent of the sequent, *e.g.,* $\wedge\mathsf{R}$, $\vee\mathsf{R}$, $\supset\mathsf{R}$, when reading the rule in a top down way. Note that both rules $\vee\mathsf{L}^{global}$ and $\supset\mathsf{L}^{global}$ have two subcases. Since a formula in the global context is true in any accessible world, to prove the succedent C, we may exploit its subformulas either in the current world, *i.e.*, the local context (rules with a subscript a), or in some accessible world, *i.e.*, some context in the frame (rules with a subscript b).

$$\frac{}{\Sigma \vdash \Delta \vdash \Gamma, p \longrightarrow p}\ \mathsf{Id}^{local} \qquad \frac{}{\Sigma \vdash \Delta, p \vdash \Gamma \longrightarrow p}\ \mathsf{Id}^{global} \qquad \frac{}{\Sigma \vdash \Delta \vdash \Gamma, \bot \longrightarrow C}\ \bot\mathsf{L}^{local}$$

$$\frac{}{\Sigma \vdash \Delta, \bot \vdash \Gamma \longrightarrow C}\ \bot\mathsf{L}^{global} \qquad \frac{}{\Sigma; \Gamma', \bot \vdash \Delta \vdash \Gamma \longrightarrow C}\ \bot\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A, B \longrightarrow C}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B \longrightarrow C}\ \wedge\mathsf{L}^{local} \qquad \frac{\Sigma \vdash \Delta, A, B \vdash \Gamma \longrightarrow C}{\Sigma \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow C}\ \wedge\mathsf{L}^{global}$$

$$\frac{\Sigma; \Gamma', A, B \vdash \Delta \vdash \Gamma \longrightarrow C}{\Sigma; \Gamma', A \wedge B \vdash \Delta \vdash \Gamma \longrightarrow C}\ \wedge\mathsf{L}^{frame} \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \qquad \Sigma \vdash \Delta \vdash \Gamma \longrightarrow B}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \wedge B}\ \wedge\mathsf{R}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow C \qquad \Sigma \vdash \Delta \vdash \Gamma, B \longrightarrow C}{\Sigma \vdash \Delta \vdash \Gamma, A \vee B \longrightarrow C}\ \vee\mathsf{L}^{local}$$

$$\frac{\Sigma \vdash \Delta, A \vee B \vdash \Gamma, A \longrightarrow C \qquad \Sigma \vdash \Delta, A \vee B \vdash \Gamma, B \longrightarrow C}{\Sigma \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow C}\ \vee\mathsf{L}_a^{global}$$

$$\frac{\Sigma; \Gamma', A \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow C \qquad \Sigma; \Gamma', B \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow C}{\Sigma; \Gamma' \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow C}\ \vee\mathsf{L}_b^{global}$$

$$\frac{\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C \qquad \Sigma; \Gamma', B \vdash \Delta \vdash \Gamma \longrightarrow C}{\Sigma; \Gamma', A \vee B \vdash \Delta \vdash \Gamma \longrightarrow C}\ \vee\mathsf{L}^{frame} \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \vee B}\ \vee\mathsf{R}_l$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow B}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \vee B}\ \vee\mathsf{R}_r \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma, A \supset B \longrightarrow A \qquad \Sigma \vdash \Delta \vdash \Gamma, B \longrightarrow C}{\Sigma \vdash \Delta \vdash \Gamma, A \supset B \longrightarrow C}\ \supset\mathsf{L}^{local}$$

$$\frac{\Sigma \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow A \qquad \Sigma \vdash \Delta, A \supset B \vdash \Gamma, B \longrightarrow C}{\Sigma \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow C}\ \supset\mathsf{L}_a^{global}$$

$$\frac{\Sigma; \Gamma \vdash \Delta, A \supset B \vdash \Gamma' \longrightarrow A \qquad \Sigma; \Gamma', B \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow C}{\Sigma; \Gamma' \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow C}\ \supset\mathsf{L}_b^{global}$$

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma', A \supset B \longrightarrow A \qquad \Sigma; \Gamma', B \vdash \Delta \vdash \Gamma \longrightarrow C}{\Sigma; \Gamma', A \supset B \vdash \Delta \vdash \Gamma \longrightarrow C}\ \supset\mathsf{L}^{frame} \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow B}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \supset B}\ \supset\mathsf{R}$$

$$\frac{\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C}{\Sigma \vdash \Delta \vdash \Gamma, \Box A \longrightarrow C}\ \Box\mathsf{L}^{local} \qquad \frac{\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C}{\Sigma \vdash \Delta, \Box A \vdash \Gamma \longrightarrow C}\ \Box\mathsf{L}^{global}$$

$$\frac{\Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow C}{\Sigma; \Gamma', \Box A \vdash \Delta \vdash \Gamma \longrightarrow C}\ \Box\mathsf{L}^{frame} \qquad \frac{\Sigma; \Gamma \vdash \Delta \vdash \cdot \longrightarrow A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow \Box A}\ \Box\mathsf{R} \qquad \frac{\Sigma; A \vdash \Delta \vdash \Gamma \longrightarrow C}{\Sigma \vdash \Delta \vdash \Gamma, \Diamond A \longrightarrow C}\ \Diamond\mathsf{L}^{local}$$

$$\frac{\Sigma; A \vdash \Delta \vdash \Gamma \longrightarrow C}{\Sigma \vdash \Delta, \Diamond A \vdash \Gamma \longrightarrow C}\ \Diamond\mathsf{L}^{global} \qquad \frac{\Sigma; \Gamma'; A \vdash \Delta \vdash \Gamma \longrightarrow C}{\Sigma; \Gamma', \Diamond A \vdash \Delta \vdash \Gamma \longrightarrow C}\ \Diamond\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow \Diamond A}\ \Diamond\mathsf{R}_a \qquad \frac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow A}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow \Diamond A}\ \Diamond\mathsf{R}_b$$

Fig. 1. Sequent Calculus $\mathbf{G_{IS5}}$

As for modal operators, $\Box A$ is provable in the current world if A is provable in every accessible world. In other words, if A is true under the local context with no assumptions, then A is true under any context and thus $\Box A$ is true. This fact is internalized by the rule $\Box$R. Conversely, if $\Box A$ is true in some context, then A is true in any context and thus we simply add A into the global context in the left rules for $\Box$. As for $\Diamond A$, it is provable if A is provable in some accessible world. Rules $\Diamond \mathsf{R}_a$ and $\Diamond \mathsf{R}_b$ correspond to the cases where A is true in the current world and some other accessible world, respectively. Conversely, if $\Diamond A$ is true in some context, it means that A is true in some accessible world, but we may not know which world it is. Hence the left rules for $\Diamond$ simply create a new context in the frame consisting solely of A.

Below, to illustrate our system, we give a proof of $\Box(A \supset B) \supset (\Box A \supset \Box B)$ in $\mathbf{G_{IS5}}$ where A and B are atomic propositions.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot \vdash A \supset B, A \vdash \cdot \longrightarrow A}\ \mathsf{Id}^{global} \quad \dfrac{}{\cdot \vdash A \supset B, A \vdash B \longrightarrow B}\ \mathsf{Id}^{local}}{\cdot \vdash A \supset B, A \vdash \cdot \longrightarrow B}\ \supset \mathsf{L}_a^{global}}{\cdot \vdash A \supset B, A \vdash \cdot \longrightarrow \Box B}\ \Box\mathsf{R}}{\cdot \vdash A \supset B \vdash \Box A \longrightarrow \Box B}\ \Box\mathsf{L}^{local}}{\cdot \vdash \cdot \vdash \Box(A \supset B), \Box A \longrightarrow \Box B}\ \Box\mathsf{L}^{local}}{\cdot \vdash \cdot \vdash \Box(A \supset B) \longrightarrow \Box A \supset \Box B}\ \supset\mathsf{R}}{\cdot \vdash \cdot \vdash \cdot \longrightarrow \Box(A \supset B) \supset (\Box A \supset \Box B)}\ \supset\mathsf{R}$$

As another example, we also give a proof of the characteristic axiom $\Diamond A \supset \Box\Diamond A$ of modal logic **S5** where A is atomic.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot \vdash \cdot \vdash A \longrightarrow A}\ \mathsf{Id}^{local}}{A \vdash \cdot \vdash \cdot \longrightarrow \Diamond A}\ \Diamond\mathsf{R}_b}{A \vdash \cdot \vdash \cdot \longrightarrow \Box\Diamond A}\ \Box\mathsf{R}}{\cdot \vdash \cdot \vdash \Diamond A \longrightarrow \Box\Diamond A}\ \Diamond\mathsf{L}^{local}}{\cdot \vdash \cdot \vdash \cdot \longrightarrow \Diamond A \supset \Box\Diamond A}\ \supset\mathsf{R}$$

2.3 Structural Properties

To prove the cut elimination theorem for $\mathbf{G_{IS5}}$, which is fundamental to the soundness of $\mathbf{G_{IS5}}$, we first need to prove various structural properties such as weakening and contraction. As contexts and frames are defined as multisets, we consider that the exchange rule is built-in. Since the cut elimination theorem is proved by induction on the size of the derivation tree, the structural properties also need to be stated and proved in terms of the size of the derivation. For this purpose, we introduce two new forms of sequents $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ and $n \geq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ where n denotes the size of the derivation tree of the sequent $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$. The difference between the two sequents is that whereas $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ means that the size of the derivation is exactly n, $n \geq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ means that the size of the derivation is less than or equal to n. In other words, $n \geq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ if and only if $m \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ for some m such that $m \leq n$. Inference rules for $\mathbf{G_{IS5}}$ with explicit proof size annotations are given in Figure 6 in Appendix. They are simply obtained by replacing $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ in the inference rules in Figure 1 with $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$, with the condition that the size of the conclusion is the sum of the sizes of the premises plus 1. The sequent calculi with and without explicit proof size annotations are equivalent as stated below.

THEOREM 2.1. *$\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ if and only if $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ for some n.*

PROOF. ($\Longrightarrow$) By induction on a derivation of $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$. ($\Longleftarrow$) By induction on a derivation of $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$. $\Box$

The structural properties that we prove below are all *size-preserving* in the sense that if a property states "$\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ implies $\Sigma' \vdash \Delta' \vdash \Gamma' \longrightarrow C'$", then it is also true that "$n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ implies $n \geq \Sigma' \vdash \Delta' \vdash \Gamma' \longrightarrow C'$", which is crucial for proving the cut elimination theorem. We first show the weakening property in Proposition 2.2 where the cases (1)–(3) allow weakening on a formula, while (4) allows us to add an arbitrary context into the frame.

Proposition 2.2 (Weakening).

(1) If $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow C$.
(2) If $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C$.
(3) If $n \simeq \Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C$.
(4) If $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow C$.

Proof. (1) and (3) are proved by simultaneous induction on n. (2) and (4) are proved by induction on n. □

In order to prove the contraction property, we need to prove the inversion property. An inference rule is said to be invertible if the conclusion sequent implies the premise sequents. Although some inference rules are not invertible, *e.g.* the left rules for $\supset$, we can still prove similar properties. Below we only show the inversion property for $\wedge$. The inversion and related properties for other connectives are given in Appendix A.

Proposition 2.3 (Inversion for $\wedge$).

(1) If $n \simeq \Sigma \vdash \Delta \vdash \Gamma, A \wedge B \longrightarrow C$ then $n \geq \Sigma \vdash \Delta \vdash \Gamma, A, B \longrightarrow C$.
(2) If $n \simeq \Sigma \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma \vdash \Delta, A, B \vdash \Gamma \longrightarrow C$.
(3) If $n \simeq \Sigma; \Gamma', A \wedge B \vdash \Delta \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma; \Gamma', A, B \vdash \Delta \vdash \Gamma \longrightarrow C$.
(4) If $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \wedge B$ then $n \geq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A$ and $n \geq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow B$.

Proof. (1) and (3) are proved by simultaneous induction on n. (2) and (4) are proved by induction on n. □

Proposition 2.4 (Contraction).

(1) If $n \simeq \Sigma \vdash \Delta \vdash \Gamma, A, A \longrightarrow C$ then $n \geq \Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow C$.
(2) If $n \simeq \Sigma \vdash \Delta, A, A \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C$.
(3) If $n \simeq \Sigma; \Gamma', A, A \vdash \Delta \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C$.
(4) If $n \simeq \Sigma \vdash \Delta, A \vdash \Gamma, A \longrightarrow C$ then $n \geq \Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C$.
(5) If $n \simeq \Sigma; \Gamma', A \vdash \Delta, A \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow C$.
(6) If $n \simeq \Sigma; \Gamma \vdash \Delta \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$.
(7) If $n \simeq \Sigma; \Gamma'; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow C$ then $n \geq \Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow C$.

Proof. (1)–(3), (6), and (7) are proved by simultaneous induction on n. (4) and (5) are proved by simultaneous induction on n. □

The contraction property basically allows us to dispense with a redundant formula in the same context. Moreover, the cases (4) and (5) permit to dispense with a formula A in the local context and some context in the frame if A is also available in the global context. Note that A in the global context is stronger than the same A in the local context and any other context in the frame. Lastly, the cases (6) and (7) permit to dispense with a context in the frame if it is the same as the local context, or there is another context in the frame that consists of exactly the same formulas. Note that in (6) we cannot eliminate the local context Γ, since it is stronger than Γ in the frame.

2.4 The Cut Elimination Theorem

In this section, we prove the admissibility of the cut rules and obtain as a corollary the cut elimination theorem. It is important not only for showing that the sequent calculus is consistent, *i.e.*, $\bot$ is not provable, but also for designing an automated theorem prover. In general, a cut rule is a lemma rule which states that if we can prove a *cut* formula A, which may be chosen arbitrarily, then we may use it as a hypothesis in proving another formula C. Knowing that the cut rules are not essential allows us to consider only the subformulas of the query formula in proof search. Since we use three kinds of contexts, we need three cut rules. Accordingly we state the admissibility of the cut rules with three clauses. Its proof uses the weakening, contraction, and inversion properties.

Theorem 2.5 (Admissibility of the cut rules).

(1) If $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A$ *and* $\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow C$ *then* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$.
(2) If $\Sigma; \Gamma \vdash \Delta \vdash \cdot \longrightarrow A$ *and* $\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C$ *then* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$.
(3) If $\Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow A$ *and* $\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C$ *then* $\Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow C$.

The second clause uses an empty local context in $\Sigma; \Gamma \vdash \Delta \vdash \cdot \longrightarrow A$ in order to prove that A is true in every accessible world. In the third clause, the current world described by Γ' in $\Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow A$ is the same as the accessible world described by Γ', A in $\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C$. We prove the three clauses simultaneously where each clause is proved by nested induction on the structure of the cut formula A and the size of the two given derivations. The proof of Theorem 2.5 and the related structural properties have been formalized in the Coq proof assistant [17].

One of the consequences of Theorem 2.5 is that the following cut rules are redundant.

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \qquad \Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow C}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C}\ \mathsf{cut}^{local}$$

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \cdot \longrightarrow A \qquad \Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C}\ \mathsf{cut}^{global}$$

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow A \qquad \Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow C}\ \mathsf{cut}^{frame}$$

In other words, any sequent in $\mathbf{G_{IS5}}$ that is provable using the above cut rules is also provable without using them. As a corollary, we obtain the following subformula property.

Corollary 2.6 (Subformula property). *Every formula in any cut-free* $\mathbf{G_{IS5}}$ *derivation of a sequent* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ *is a subformula of* Σ, Δ, Γ, *or* C.

Although $\mathbf{G_{IS5}}$ may be used as a basis for backward proof search thanks to the subformula property, to achieve a decision procedure, we may need to implement an efficient history-based loop-detection mechanism such as one for **IS4** presented in [12]. Instead of pursuing this direction, we will focus on forward proof search in the next section, and how to combine forward and backward proof search to perform efficient bidirectional proof search. Before finishing this section, we show that $\mathbf{G_{IS5}}$ is sound and complete with respect to the Kripke semantics of **IS5**, that is, any valid formula in **IS5** has a proof in $\mathbf{G_{IS5}}$ and any formula provable in $\mathbf{G_{IS5}}$ is valid in **IS5**.

Theorem 2.7 (Soundness).
If $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ *has a proof in* $\mathbf{G_{IS5}}$ *then it is valid in* **IS5**.

Theorem 2.8 (Completeness).
If A *is valid in* **IS5** *then* $\cdot \vdash \cdot \vdash \cdot \longrightarrow A$ *has a proof in* $\mathbf{G_{IS5}}$.

Instead of directly proving Theorems 2.7 and 2.8, we prove that $\mathbf{G_{IS5}}$ is equivalent to the MC-sequent calculus proposed in [8], which is shown to be sound and complete with respect to **IS5**. We present the detailed proof in Appendix B.

3 A SEQUENT CALCULUS FOR FORWARD PROOF SEARCH

In this section, we present a focused sequent calculus based on focusing [1] and the signed subformula property, which is suitable for forward proof search and equivalent to $\mathbf{G_{IS5}}$.

3.1 A Focused Sequent Calculus

We first adopt the well-known focusing technique for forward proof search to our sequent calculus. When a sequent has a focused formula A, the only applicable rules are those with A as a principal formula. The use of global contexts necessitates two kinds of focusing mechanisms, denoted by $\rhd$ and $\rhd\rhd$, respectively [10]. We use the notation $\rhd A$ to mean that A is assumed true in the current world or some accessible world, and $\rhd \rhd A$ to mean that A is assumed valid, *i.e.*, true in every accessible world. The use of frames doubles the number of focused sequents we need, and thus we use the following five sequents for our focused sequent calculus $\mathbf{G^F_{IS5}}$.

- $\Sigma \vdash \Delta \vdash \Gamma \longmapsto C$ means that C can be proved using all assumptions in Σ, Δ, Γ.
- $\Sigma \vdash \Delta \vdash \Gamma \rhd A \longmapsto C$ means that C can be proved using all assumptions in Σ, Δ, Γ, with A assumed true in the current world described by Γ.
- $\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd A \longmapsto C$ means that C can be proved using all assumptions in Σ, Δ, Γ, with A assumed valid and being analyzed in the current world described by Γ.
- $\Sigma; \Gamma' \rhd A \vdash \Delta \vdash \Gamma \longmapsto C$ means that C can be proved using all assumptions in Σ, Γ', Δ, Γ, with A assumed true at some accessible world described by Γ'.
- $\Sigma; \Gamma' \rhd \rhd A \vdash \Delta \vdash \Gamma \longmapsto C$ means that C can be proved using all assumptions in Σ, Γ', Δ, Γ, with A assumed valid and being analyzed in some accessible world described by Γ'.

Following Girard [9], we call the position of a focused formula in a focused sequent the *stoup*. In particular, $\rhd \rhd A$ is called the *modal stoup*.

Figures 2 and 3 show the inference rules for $\mathbf{G^F_{IS5}}$, which are obtained from those for $\mathbf{G_{IS5}}$. The main difference between $\mathbf{G_{IS5}}$ and $\mathbf{G^F_{IS5}}$ is that in a $\mathbf{G^F_{IS5}}$ sequent all assumptions in the contexts, and a focused formula if any, are needed to prove the consequent of the sequent. Moreover, a context is considered to be a set rather than a multiset, *i.e.*, a duplicate assumption is not allowed. Hence, in the $\mathbf{G^F_{IS5}}$ system, we write Γ_1, Γ_2 for a set union $\Gamma_1 \cup \Gamma_2$ and Γ, A for $\Gamma \cup \{A\}$. Under this interpretation, the axioms such as the Id and $\bot\mathsf{L}$ rules now use an empty context, and the contexts of the conclusion sequent of a $\mathbf{G^F_{IS5}}$ inference rule are the union of the contexts of its premises. Furthermore, since general weakening is not allowed in $\mathbf{G^F_{IS5}}$, we introduce local weakening in the rule $\supset\mathsf{R}_b$. Except for the interpretation of contexts and the rule $\supset\mathsf{R}_b$, the right rules in $\mathbf{G^F_{IS5}}$ are the same as those in $\mathbf{G_{IS5}}$.

Another difference between $\mathbf{G_{IS5}}$ and $\mathbf{G^F_{IS5}}$ is that the principal formula, say A, in the conclusion of every $\mathbf{G^F_{IS5}}$ left rule is now moved into the corresponding stoup, that is, either $\rhd A$ or $\rhd \rhd A$. As for the premises, either a subformula of A is analyzed in the same stoup as the conclusion (rules $\wedge\mathsf{L}$ and $\supset\mathsf{L}$), or the premises have no focused formula (rules $\vee\mathsf{L}$, $\Box\mathsf{L}$, and $\Diamond\mathsf{L}$). In the former case, the focused premise sequent must be proved by a left rule, which is determined by its focused formula, whereas in the latter case, the premises must be proved by a right rule or one of the ch rules, which are newly introduced in $\mathbf{G^F_{IS5}}$. When constructing a proof tree in $\mathbf{G^F_{IS5}}$ in a forward direction, we begin at the leaves with either Id or $\bot\mathsf{L}$ rules, which are followed by a sequence of left-rule applications. Then, before we apply a right rule or a left rule whose premise has no focus, one of the ch rules should be applied to drop the stoup formula into the corresponding context.

$$\frac{}{\cdot \vdash \cdot \vdash \cdot \rhd p \longmapsto p}\ \mathsf{Id}^{local} \qquad \frac{}{\cdot \vdash \cdot \vdash \cdot \rhd \rhd p \longmapsto p}\ \mathsf{Id}^{global} \qquad \frac{}{\cdot \vdash \cdot \vdash \cdot \rhd \bot \longmapsto C}\ \bot\mathsf{L}^{local}$$

$$\frac{}{\cdot \vdash \cdot \vdash \cdot \rhd \rhd \bot \longmapsto C}\ \bot\mathsf{L}_a^{global} \qquad \frac{}{\cdot \rhd \rhd \bot \vdash \cdot \vdash \cdot \longmapsto C}\ \bot\mathsf{L}_b^{global}$$

$$\frac{}{\cdot \rhd \bot \vdash \cdot \vdash \cdot \longmapsto C}\ \bot\mathsf{L}^{frame} \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma \rhd A \longmapsto C}{\Sigma \vdash \Delta \vdash \Gamma, A \longmapsto C}\ \mathsf{ch}^{local}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd A \longmapsto C}{\Sigma \vdash \Delta, A \vdash \Gamma \longmapsto C}\ \mathsf{ch}_a^{global} \qquad \frac{\Sigma; \Gamma_1 \rhd \rhd A \vdash \Delta \vdash \Gamma_2 \longmapsto C}{\Sigma; \Gamma_1 \vdash \Delta, A \vdash \Gamma_2 \longmapsto C}\ \mathsf{ch}_b^{global}$$

$$\frac{\Sigma; \Gamma_1 \rhd A \vdash \Delta \vdash \Gamma_2 \longmapsto C}{\Sigma; \Gamma_1, A \vdash \Delta \vdash \Gamma_2 \longmapsto C}\ \mathsf{ch}^{frame} \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma \rhd A_i \longmapsto C}{\Sigma \vdash \Delta \vdash \Gamma \rhd A_1 \wedge A_2 \longmapsto C}\ \wedge\mathsf{L}^{local}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd A_i \longmapsto C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd A_1 \wedge A_2 \longmapsto C}\ \wedge\mathsf{L}_a^{global} \qquad \frac{\Sigma; \Gamma_1 \rhd \rhd A_i \vdash \Delta \vdash \Gamma_2 \longmapsto C}{\Sigma; \Gamma_1 \rhd \rhd A_1 \wedge A_2 \vdash \Delta \vdash \Gamma_2 \longmapsto C}\ \wedge\mathsf{L}_b^{global}$$

$$\frac{\Sigma; \Gamma_1 \rhd A_i \vdash \Delta \vdash \Gamma_2 \longmapsto C}{\Sigma; \Gamma_1 \rhd A_1 \wedge A_2 \vdash \Delta \vdash \Gamma_2 \longmapsto C}\ \wedge\mathsf{L}^{frame} \qquad \frac{\Sigma_1 \vdash \Delta_1 \vdash \Gamma_1 \longmapsto A_1 \qquad \Sigma_2 \vdash \Delta_2 \vdash \Gamma_2 \longmapsto A_2}{\Sigma_1; \Sigma_2 \vdash \Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2 \longmapsto A_1 \wedge A_2}\ \wedge\mathsf{R}$$

$$\frac{\Sigma_1 \vdash \Delta_1 \vdash \Gamma_1, A_1 \longmapsto C \qquad \Sigma_2 \vdash \Delta_2 \vdash \Gamma_2, A_2 \longmapsto C}{\Sigma_1; \Sigma_2 \vdash \Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2 \rhd A_1 \vee A_2 \longmapsto C}\ \vee\mathsf{L}^{local}$$

$$\frac{\Sigma_1 \vdash \Delta_1 \vdash \Gamma_1, A_1 \longmapsto C \qquad \Sigma_2 \vdash \Delta_2 \vdash \Gamma_2, A_2 \longmapsto C}{\Sigma_1; \Sigma_2 \vdash \Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2 \rhd \rhd A_1 \vee A_2 \longmapsto C}\ \vee\mathsf{L}_a^{global}$$

$$\frac{\Sigma_1; \Gamma_1, A_1 \vdash \Delta_1 \vdash \Gamma_3 \longmapsto C \qquad \Sigma_2; \Gamma_2, A_2 \vdash \Delta_2 \vdash \Gamma_4 \longmapsto C}{\Sigma_1; \Sigma_2; \Gamma_1, \Gamma_2 \rhd \rhd A_1 \vee A_2 \vdash \Delta_1, \Delta_2 \vdash \Gamma_3, \Gamma_4 \longmapsto C}\ \vee\mathsf{L}_b^{global}$$

$$\frac{\Sigma_1; \Gamma_1, A_1 \vdash \Delta_1 \vdash \Gamma_3 \longmapsto C \qquad \Sigma_2; \Gamma_2, A_2 \vdash \Delta_2 \vdash \Gamma_4 \longmapsto C}{\Sigma_1; \Sigma_2; \Gamma_1, \Gamma_2 \rhd A_1 \vee A_2 \vdash \Delta_1, \Delta_2 \vdash \Gamma_3, \Gamma_4 \longmapsto C}\ \vee\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longmapsto A_1}{\Sigma \vdash \Delta \vdash \Gamma \longmapsto A_1 \vee A_2}\ \vee\mathsf{R}_l \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma \longmapsto A_2}{\Sigma \vdash \Delta \vdash \Gamma \longmapsto A_1 \vee A_2}\ \vee\mathsf{R}_r$$

$$i \in \{1, 2\}$$

Fig. 2. Focused Sequent Calculus $\mathbf{G}^{\mathsf{F}}_{\mathsf{IS5}}$ for Forward Proof Search

3.2 Signed Subformulas

Together with focusing, we can significantly reduce the search space by classifying the subformulas of a query formula into positive and negative classes as in the standard inverse method for forward proof search [6]. The sign of a subformula, indicated by using a superscript such as + and −, determines in which place of a sequent the subformula may occur during proof search. Following [10], we also use five types of signs as follows:

- Positive (+) subformulas may occur as goal formulas.
- Negative (−) subformulas may occur in the local context or a context in the frame.
- Negative focused (∼) subformulas may occur in the nonmodal stoup, either in the local context or in the frame.
- Valid (=) subformulas may occur in the global context.

$$\frac{\Sigma_1 \vdash \Delta_1 \vdash \Gamma_1 \longmapsto A_1 \qquad \Sigma_2 \vdash \Delta_2 \vdash \Gamma_2 \rhd A_2 \longmapsto C}{\Sigma_1; \Sigma_2 \vdash \Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2 \rhd A_1 \supset A_2 \longmapsto C} \supset\mathsf{L}^{local}$$

$$\frac{\Sigma_1 \vdash \Delta_1 \vdash \Gamma_1 \longmapsto A_1 \qquad \Sigma_2 \vdash \Delta_2 \vdash \Gamma_2 \rhd \rhd A_2 \longmapsto C}{\Sigma_1; \Sigma_2 \vdash \Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2 \rhd \rhd A_1 \supset A_2 \longmapsto C} \supset\mathsf{L}_a^{global}$$

$$\frac{\Sigma_1; \Gamma_3 \vdash \Delta_1 \vdash \Gamma_1 \longmapsto A_1 \qquad \Sigma_2; \Gamma_2 \rhd \rhd A_2 \vdash \Delta_2 \vdash \Gamma_4 \longmapsto C}{\Sigma_1; \Sigma_2; \Gamma_1, \Gamma_2 \rhd \rhd A_1 \supset A_2 \vdash \Delta_1, \Delta_2 \vdash \Gamma_3, \Gamma_4 \longmapsto C} \supset\mathsf{L}_b^{global}$$

$$\frac{\Sigma_1; \Gamma_3 \vdash \Delta_1 \vdash \Gamma_1 \longmapsto A_1 \qquad \Sigma_2; \Gamma_2 \rhd A_2 \vdash \Delta_2 \vdash \Gamma_4 \longmapsto C}{\Sigma_1; \Sigma_2; \Gamma_1, \Gamma_2 \rhd A_1 \supset A_2 \vdash \Delta_1, \Delta_2 \vdash \Gamma_3, \Gamma_4 \longmapsto C} \supset\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A_1 \longmapsto A_2}{\Sigma \vdash \Delta \vdash \Gamma \longmapsto A_1 \supset A_2} \supset\mathsf{R}_a \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma \longmapsto A_2}{\Sigma \vdash \Delta \vdash \Gamma \longmapsto A_1 \supset A_2} \supset\mathsf{R}_b$$

$$\frac{\Sigma \vdash \Delta, A \vdash \Gamma \longmapsto C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \Box A \longmapsto C} \Box\mathsf{L}^{local} \qquad \frac{\Sigma \vdash \Delta, A \vdash \Gamma \longmapsto C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd \Box A \longmapsto C} \Box\mathsf{L}_a^{global}$$

$$\frac{\Sigma; \Gamma_1 \vdash \Delta, A \vdash \Gamma_2 \longmapsto C}{\Sigma; \Gamma_1 \rhd \rhd \Box A \vdash \Delta \vdash \Gamma_2 \longmapsto C} \Box\mathsf{L}_b^{global} \qquad \frac{\Sigma; \Gamma_1 \vdash \Delta, A \vdash \Gamma_2 \longmapsto C}{\Sigma; \Gamma_1 \rhd \Box A \vdash \Delta \vdash \Gamma_2 \longmapsto C} \Box\mathsf{L}^{frame}$$

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \cdot \longmapsto A}{\Sigma \vdash \Delta \vdash \Gamma \longmapsto \Box A} \Box\mathsf{R} \qquad \frac{\Sigma; A \vdash \Delta \vdash \Gamma \longmapsto C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \Diamond A \longmapsto C} \Diamond\mathsf{L}^{local}$$

$$\frac{\Sigma; A \vdash \Delta \vdash \Gamma \longmapsto C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd \Diamond A \longmapsto C} \Diamond\mathsf{L}_a^{global} \qquad \frac{\Sigma; \Gamma_1; A \vdash \Delta \vdash \Gamma_2 \longmapsto C}{\Sigma; \Gamma_1 \rhd \rhd \Diamond A \vdash \Delta \vdash \Gamma_2 \longmapsto C} \Diamond\mathsf{L}_b^{global}$$

$$\frac{\Sigma; \Gamma_1; A \vdash \Delta \vdash \Gamma_2 \longmapsto C}{\Sigma; \Gamma_1 \rhd \Diamond A \vdash \Delta \vdash \Gamma_2 \longmapsto C} \Diamond\mathsf{L}^{frame} \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma \longmapsto A}{\Sigma \vdash \Delta \vdash \Gamma \longmapsto \Diamond A} \Diamond\mathsf{R}_a \qquad \frac{\Sigma; \Gamma_2 \vdash \Delta \vdash \Gamma_1 \longmapsto A}{\Sigma; \Gamma_1 \vdash \Delta \vdash \Gamma_2 \longmapsto \Diamond A} \Diamond\mathsf{R}_b$$

Fig. 3. Focused Sequent Calculus $\mathbf{G}^{\mathsf{F}}_{\mathbf{IS5}}$ for Forward Proof Search (Continued)

- Valid focused (≈) subformulas may occur in the modal stoup, either in the local context or in the frame.

With this interpretation, the following signed subformula relation is derived from the inference rules for $\mathbf{G}^{\mathsf{F}}_{\mathbf{IS5}}$.

Definition 3.1 (Signed subformulas). A signed subformula A^* is a formula A with a sign $* \in \{+, -, \sim, =, \approx\}$. The signed subformula relation $\leq$ is defined as the smallest reflexive, transitive relation satisfying the following:

$$A_1^+, A_2^+ \leq (A_1 \wedge A_2)^+ \qquad A_1^+, A_2^+ \leq (A_1 \vee A_2)^+ \qquad A_1^-, A_2^+ \leq (A_1 \supset A_2)^+$$

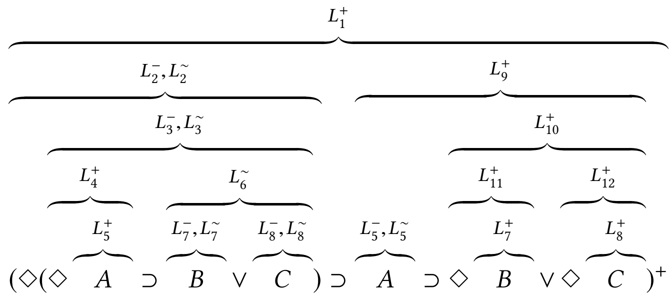
$$A^+ \leq (\Box A)^+ \qquad A^+ \leq (\Diamond A)^+ \qquad A^\sim \leq A^-$$

$$A_1^\sim, A_2^\sim \leq (A_1 \wedge A_2)^\sim \qquad A_1^-, A_2^- \leq (A_1 \vee A_2)^\sim \qquad A_1^+, A_2^\sim \leq (A_1 \supset A_2)^\sim$$

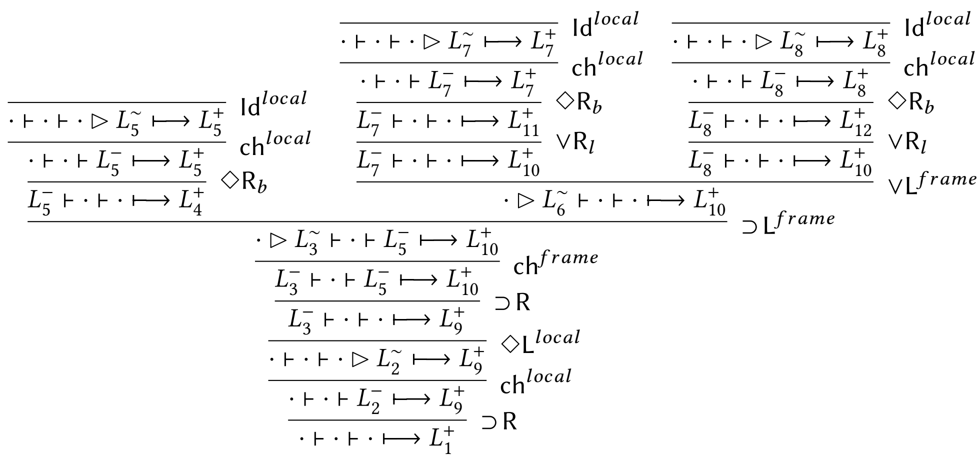
$$A^= \leq (\Box A)^\sim \qquad A^- \leq (\Diamond A)^\sim \qquad A^\approx \leq A^=$$

$$A_1^\approx, A_2^\approx \leq (A_1 \wedge A_2)^\approx \qquad A_1^-, A_2^- \leq (A_1 \vee A_2)^\approx \qquad A_1^+, A_2^\approx \leq (A_1 \supset A_2)^\approx$$

$$A^= \leq (\Box A)^\approx \qquad A^- \leq (\Diamond A)^\approx$$



(a) An Example of a Query Formula and its Signed Subformulas (A, B, C are atomic)



(b) A Signed Derivation Tree for L_1^+

Fig. 4. An Example of Forward Proof Search

Based on the above interpretation of signed subformulas, we can annotate with signs the inference rules in Figures 2 and 3 in a consistent way. Such a signed version of $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ is given in Appendix C. Figure 4 shows an example of a query formula, its signed subformulas, and its signed derivation tree. For simplicity, we assign a unique numbered label to each subformula; if two subformulas are identical, they are given a same label. Note that only the signed subformulas of L_1^+ are used in its signed derivation in Figure 4b. In other words, when constructing a derivation tree for L_1^+, we do not need to consider the formulas such as $A^=$ that are not a signed subformula of L_1^+. In general, during proof search for a given query formula in $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$, we only need to consider its signed subformulas, as stated in Theorem 3.2. Below we use Γ^- and $\Delta^=$ to denote contexts of signed subformulas of the forms $A_1^-, \ldots, A_n^-$ and $B_1^=, \ldots, B_m^=$, respectively. We also write Σ^- for a frame of signed contexts of the form $\Gamma_1^-; \ldots; \Gamma_l^-$.

Theorem 3.2 (Signed subformula property). *Every sequent in a signed $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ derivation of*

- $\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+$,
- $\Sigma^- \vdash \Delta^= \vdash \Gamma^- \rhd A^{\sim} \longmapsto C^+$,
- $\Sigma^- \vdash \Delta^= \vdash \Gamma^- \rhd \rhd A^{\approx} \longmapsto C^+$,
- $\Sigma^-; \Gamma'^- \rhd A^{\sim} \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+$, *or*

- $\Sigma^-; \Gamma'^- \rhd \rhd A^{\approx} \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+$

is of the form

- $\Gamma_1^-; \ldots; \Gamma_l^- \vdash D_1^=, \ldots, D_m^= \vdash E_1^-, \ldots, E_n^- \longmapsto F^+$,
- $\Gamma_1^-; \ldots; \Gamma_l^- \vdash D_1^=, \ldots, D_m^= \vdash E_1^-, \ldots, E_n^- \rhd E^{\sim} \longmapsto F^+$,
- $\Gamma_1^-; \ldots; \Gamma_l^- \vdash D_1^=, \ldots, D_m^= \vdash E_1^-, \ldots, E_n^- \rhd \rhd E^{\approx} \longmapsto F^+$,
- $\Gamma_1^-; \ldots; \Gamma_l^- \rhd E^{\sim} \vdash D_1^=, \ldots, D_m^= \vdash E_1^-, \ldots, E_n^- \longmapsto F^+$, *or*
- $\Gamma_1^-; \ldots; \Gamma_l^- \rhd \rhd E^{\approx} \vdash D_1^=, \ldots, D_m^= \vdash E_1^-, \ldots, E_n^- \longmapsto F^+$

where all formulas H_{ij} in Γ_i^-, all $D_t^=$, all E_u^-, $E^{\sim}$, $E^{\approx}$, and F^+ are signed subformulas of Σ^-, $\Delta^=$, Γ^-, Γ'^-, $A^{\sim}$, $A^{\approx}$, and C^+.

Proof. By simultaneous induction on the structure of the given derivations. □

Based on the signed subformula property, forward proof search for $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ may proceed as follows. First, we enumerate every signed subformula of a given query formula. Second, we generate every valid, initial signed sequent by applying the axiom Id or $\perp$L on available signed subformulas. For example, in Figure 4b, using the rule Id^{local}, we generate three initial sequents $\cdot \vdash \cdot \vdash \cdot \rhd L_5^{\sim} \longmapsto L_5^+$, $\cdot \vdash \cdot \vdash \cdot \rhd L_7^{\sim} \longmapsto L_7^+$, and $\cdot \vdash \cdot \vdash \cdot \rhd L_8^{\sim} \longmapsto L_8^+$. Third, we generate new valid sequents by applying inference rules on currently available valid sequents, *i.e.*, using them as premises. The conclusion sequent of each rule application is valid only if it consists of signed subformulas of the query formula. For example, in Figure 4b, we do not need to apply the rule ch^{frame} on a valid sequent $\cdot \rhd L_6^{\sim} \vdash \cdot \vdash \cdot \longmapsto L_{10}^+$ since its conclusion sequent $L_6^- \vdash \cdot \vdash \cdot \longmapsto L_{10}^+$ contains L_6^- which is not a signed subformula of the query formula L_1^+. Next, we repeat the third step until no more new valid sequents are generated. This process eventually terminates since there are only a finite number of signed subformulas and thus valid signed sequents for the query formula. Lastly, if the query formula is contained in the final set of generated, valid sequents, then it is provable and we can construct its signed derivation tree. Otherwise, it is not provable.

Although focusing and the signed subformula property enable us to implement an efficient forward proof search procedure, there is still room for improvement. In particular, for each unfocused valid sequent generated by forward proof search, we may need to consider each right rule application to see if a new valid conclusion sequent is generated. In contrast, in backward proof search, the structure of the goal formula determines which right rules can be applied. In Section 4, we discuss how to combine forward and backward proof search to take advantage of both strategies.

3.3 Soundness and Completeness

Before moving on to the next section, we show the equivalence of $\mathbf{G}_{\mathbf{IS5}}$ and $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$, that is, any sequent in $\mathbf{G}_{\mathbf{IS5}}$ is provable if and only if it is provable in $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$.

Theorem 3.3 (Soundness of $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ with respect to $\mathbf{G}_{\mathbf{IS5}}$).
(1) If $\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+$ then $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$.
(2) If $\Sigma^- \vdash \Delta^= \vdash \Gamma^- \rhd A^{\sim} \longmapsto C^+$ then $\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow C$.
(3) If $\Sigma^- \vdash \Delta^= \vdash \Gamma^- \rhd \rhd A^{\approx} \longmapsto C^+$ then $\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C$.
(4) If $\Sigma^-; \Gamma'^- \rhd A^{\sim} \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+$ then $\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C$.
(5) If $\Sigma^-; \Gamma'^- \rhd \rhd A^{\approx} \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+$ then $\Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow C$.

Theorem 3.4 (Completeness of $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ with respect to $\mathbf{G}_{\mathbf{IS5}}$).
If $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ then $\Sigma_f^- \vdash \Delta_f^= \vdash \Gamma_f^- \longmapsto C^+$ where Σ_f, Δ_f and Γ_f are subsets of Σ, Δ and Γ, respectively.

While proving Theorem 3.3 is relatively easy because $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ derivations are a particular form of $\mathbf{G}_{\mathbf{IS5}}$ derivations, proving Theorem 3.4 is much more complex because we need to show that if a $\mathbf{G}_{\mathbf{IS5}}$ sequent has a derivation tree, then there exists a derivation tree of a particular shape where (1) every left-rule application is focused, that is, each left-rule application analyzes the formula introduced by the previous left-rule application when reading the derivation tree in a bottom-up way and (2) there is no redundant rule application which adds into the contexts formulas that are not strictly necessary. Instead of directly proving the equivalence of $\mathbf{G}_{\mathbf{IS5}}$ and $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$, we prove it using two intermediate equivalent sequent calculi, named $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ and $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$, respectively.

The first intermediate system $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ is given in Appendix D and it mainly bridges the gap between $\wedge\mathsf{L}$ rules of $\mathbf{G}_{\mathbf{IS5}}$ and $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$. $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ differs from $\mathbf{G}_{\mathbf{IS5}}$ in two aspects: (1) it keeps the principal formula in the conclusion sequent of every left rule in the premise sequents so that it may be re-analyzed, and (2) its left rules for $\wedge$ ($\wedge\mathsf{L}^{local}$, $\wedge\mathsf{L}^{global}_a$, $\wedge\mathsf{L}^{global}_b$, $\wedge\mathsf{L}^{frame}$) select only one of A_1 and A_2 from the principal formula $A_1 \wedge A_2$ and use it in the premise sequent, which is also the case for $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$. The second intermediate system $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ is given in Appendix E and it is equipped with a focusing mechanism as in $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ but it allows redundant rule applications as in $\mathbf{G}_{\mathbf{IS5}}$ (formulas in the contexts may not be used in the derivation tree). We show the equivalence of the following systems and thus establishing the equivalence of $\mathbf{G}_{\mathbf{IS5}}$ and $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ (the proofs are given in Appendices D, E, and F):

$$\mathbf{G}_{\mathbf{IS5}} \Longleftrightarrow \mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}} \Longleftrightarrow \mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}} \Longleftrightarrow \mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$$

4 BIDIRECTIONAL PROOF SEARCH FOR $\mathbf{G}_{\mathbf{IS5}}$

This section presents a bidirectional decision procedure for $\mathbf{G}_{\mathbf{IS5}}$ which consists of two steps. Given a query formula A, it first generates a set of derived inference rules that represent partial derivation trees, that may occur in a proof tree of A, consisting of only focused left-rule applications with signed subformulas of A (Section 4.1). Then, it attempts to build a proof tree composed of only those derived rules and right rules of $\mathbf{G}_{\mathbf{IS5}}$, performing backward proof search (Section 4.2). Our bidirectional decision procedure is sound and complete with respect to $\mathbf{G}_{\mathbf{IS5}}$ and is guaranteed to terminate.

4.1 Focused Threads and Derived Rules

The derived rules that our bidirectional decision procedure generates exactly correspond to the notion of *focused threads* in $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ derivations, defined as follows (adopted from [10]).

Definition 4.1 (Focused thread). A focused thread of a $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ derivation is a segment of the derivation that satisfies the following three conditions:

- First, it begins at the top with an application of either an axiom (Id or $\bot$L) or a left rule that introduces a focus in the conclusion sequent but has no focus in its premise sequents ($\vee$L, $\Box$L, or $\Diamond$L).
- Second, it includes only focused left-rule applications between the top and bottom rule applications.
- Finally, it ends with an application of one of the ch rules which move the stoup formula into one of the contexts.

With this definition, in any $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ derivation, a left-rule application may appear only inside a focused thread. In other words, any $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ derivation can be considered as consisting of only focused threads strung together by right-rule applications.

The key idea behind our bidirectional decision procedure is that given a query formula A, we can deterministically construct every focused thread that may occur in a proof tree of A using focusing and the signed subformula property, before performing actual proof search. Then, each focused thread is translated into a corresponding derived rule for $\mathbf{G_{IS5}}$, so that it can be used in backward proof search. To illustrate, let us consider the example formula L_1^+ given in Figure 4a again. We have the following signed subformulas:

$$L_1^+, L_4^+, L_5^+, L_7^+, L_8^+, L_9^+, L_{10}^+, L_{11}^+, L_{12}^+, \quad L_2^-, L_3^-, L_5^-, L_7^-, L_8^-, \quad L_2^\sim, L_3^\sim, L_5^\sim, L_6^\sim, L_7^\sim, L_8^\sim$$

To construct focused threads, we first consider the Id^{local} rule and apply it to atomic signed subformulas such as $L_5^\sim$, L_5^+, $L_7^\sim$, L_7^+, $L_8^\sim$, and L_8^+ . For example, we may generate the following partial derivation tree which is valid with respect to the signed subformula property:

$$\dfrac{}{\cdot \vdash \cdot \vdash \cdot \rhd L_5^\sim \longmapsto L_5^+}\ \mathsf{Id}^{local}$$

Next, we check if there is a left rule that may be applied to this derivation. Since L_5^- is the only immediate parent of $L_5^\sim$ in the subformula hierarchy, the only applicable rule is ch^{local} which drops the stoup formula $L_5^\sim$ into the local context:

$$\dfrac{\dfrac{}{\cdot \vdash \cdot \vdash \cdot \rhd L_5^\sim \longmapsto L_5^+}\ \mathsf{Id}^{local}}{\cdot \vdash \cdot \vdash L_5^- \longmapsto L_5^+}\ \mathsf{ch}^{local}$$

This focused thread may occur in a $\mathbf{G^F_{IS5}}$ derivation of L_1^+, from which we derive a corresponding schematic $\mathbf{G_{IS5}}$ inference rule by applying weakening and removing the signs of the subformulas as follows:

$$\dfrac{}{\Sigma \vdash \Delta \vdash \Gamma, \mathbf{L_5} \longrightarrow L_5}\ \mathsf{Id}^{local}(L_5)$$

We determine the name of a derived rule by the name of the top rule application and the principal formula of the conclusion sequent of the corresponding focused thread; for example, we name the above derived rule $\mathsf{Id}^{local}(L_5)$. In the rule $\mathsf{Id}^{local}(L_5)$, the frame Σ and the contexts Δ, Γ are schematic because a $\mathbf{G_{IS5}}$ sequent is allowed to have redundant formulas in the contexts. In contrast, L_5 is a fixed, specific formula and is not schematic.

As with $\mathsf{Id}^{local}(L_5)$, we also derive $\mathbf{G_{IS5}}$ inference rules $\mathsf{Id}^{local}(L_7)$ and $\mathsf{Id}^{local}(L_8)$ from the following $\mathbf{G^F_{IS5}}$ focused threads:

$$\dfrac{\dfrac{}{\cdot \vdash \cdot \vdash \cdot \rhd L_7^\sim \longmapsto L_7^+}\ \mathsf{Id}^{local}}{\cdot \vdash \cdot \vdash L_7^- \longmapsto L_7^+}\ \mathsf{ch}^{local} \quad \rightsquigarrow \quad \dfrac{}{\Sigma \vdash \Delta \vdash \Gamma, \mathbf{L_7} \longrightarrow L_7}\ \mathsf{Id}^{local}(L_7)$$

$$\dfrac{\dfrac{}{\cdot \vdash \cdot \vdash \cdot \rhd L_8^\sim \longmapsto L_8^+}\ \mathsf{Id}^{local}}{\cdot \vdash \cdot \vdash L_8^- \longmapsto L_8^+}\ \mathsf{ch}^{local} \quad \rightsquigarrow \quad \dfrac{}{\Sigma \vdash \Delta \vdash \Gamma, \mathbf{L_8} \longrightarrow L_8}\ \mathsf{Id}^{local}(L_8)$$

When constructing focused threads, we do not consider the Id^{global} and $\bot\mathsf{L}$ rules since L_1^+ neither includes a subformula with sign $\approx$ nor $\bot$ as a subformula.

We also generate a $\mathbf{G_{IS5}}$ derived rule from a $\mathbf{G^F_{IS5}}$ focused thread that begins with the $\vee\mathsf{L}^{local}$ rule. By the signed subformula property, we can apply the $\vee\mathsf{L}^{local}$ rule on L_7^- and L_8^- which produces a stoup formula $L_6^\sim$; then the only possible rule application is $\supset\mathsf{L}^{local}$ which produces a stoup formula $L_3^\sim$; finally from $L_3^\sim \leq L_3^-$, the last rule application should be ch^{local} which ends the focused thread. Then, by applying weakening, removing the signs, and collapsing the rule applications, we obtain a $\mathbf{G_{IS5}}$ derived rule as follows:

$$\dfrac{\Sigma_3^- \vdash \Delta_3^= \vdash \Gamma_3^- \longmapsto \mathbf{L}_4^+ \qquad \dfrac{\dfrac{\Sigma_1^- \vdash \Delta_1^= \vdash \Gamma_1^-, \mathbf{L}_7^- \longmapsto M^+ \quad \Sigma_2^- \vdash \Delta_2^= \vdash \Gamma_2^-, \mathbf{L}_8^- \longmapsto M^+}{\Sigma_1^-; \Sigma_2^- \vdash \Delta_1^=, \Delta_2^= \vdash \Gamma_1^-, \Gamma_2^- \rhd \mathbf{L}_6^{\sim} \longmapsto M^+} \vee\mathsf{L}^{local}}{}}{\dfrac{\Sigma_3^-; \Sigma_1^-; \Sigma_2^- \vdash \Delta_3^=, \Delta_1^=, \Delta_2^= \vdash \Gamma_3^-, \Gamma_1^-, \Gamma_2^- \rhd \mathbf{L}_3^{\sim} \longmapsto M^+}{\Sigma_3^-; \Sigma_1^-; \Sigma_2^- \vdash \Delta_3^=, \Delta_1^=, \Delta_2^= \vdash \Gamma_3^-, \Gamma_1^-, \Gamma_2^-, \mathbf{L}_3^- \longmapsto M^+} \mathsf{ch}^{local}} \supset\mathsf{L}^{local}$$

$$\rightsquigarrow \quad \dfrac{\overbrace{\Sigma \vdash \Delta \vdash \Gamma, L_3 \longrightarrow \mathbf{L_4}}^{twig} \quad \overbrace{\Sigma \vdash \Delta \vdash \Gamma, L_3, \mathbf{L_7} \longrightarrow M \quad \Sigma \vdash \Delta \vdash \Gamma, L_3, \mathbf{L_8} \longrightarrow M}^{trunks}}{\Sigma \vdash \Delta \vdash \Gamma, \mathbf{L_3} \longrightarrow M} \vee\mathsf{L}^{local}(L_3)$$

Note that both the focused thread and the derived rule above are parametric not only with respect to the contexts Σ, Δ, Γ but also with respect to the goal formula M. Technically, only their specific instantiations that appear in a derivation with concrete contexts and a concrete goal formula correspond to the notion of focused threads, according to Definition 4.1. To distinguish concrete focused threads and derived rules from schematic ones, we will refer to them as focused thread *instances* and derived rule *instances*, respectively.

A derived rule may have two kinds of premises: *trunk* and *twig* premises. A trunk sequent of a $\mathbf{G_{IS5}}$ derived rule is a premise of the top rule application of the corresponding $\mathbf{G^F_{IS5}}$ focused thread (*e.g.*, $\vee\mathsf{L}^{local}$ in the rule $\vee\mathsf{L}^{local}(L_3)$). It has the same goal formula (*e.g.*, M) as the conclusion sequent and contains an additional subformula (*e.g.*, L_7 or L_8) in the local context. A twig sequent is a premise of a $\supset\mathsf{L}^{local}$ rule application that occurs in the middle of the focused thread. It has the same antecedent as the conclusion sequent but with a different goal formula (*e.g.*, L_4 in the rule $\vee\mathsf{L}^{local}(L_3)$). In the backward proof search, we treat trunk and twig sequents differently to ensure its termination. We also note that the principal formula L_3 remains in the premise sequents of $\vee\mathsf{L}^{local}(L_3)$ unlike its corresponding focused thread. It makes it easier to implement our backward proof search algorithm ($\mathbf{G^K_{IS5}}$, given in Appendix E, is indeed the system that keeps the principal formula of the conclusion sequent in the premises, and we use it as the base system to generate derived rules instead of $\mathbf{G^F_{IS5}}$ for our implementation).

Another important point is that a derived rule analyzing a subformula in the local context may have a dual rule analyzing the same subformula in an accessible context in the frame. For example, the rule $\vee\mathsf{L}^{local}(L_3)$ has the following dual rule:

$$\dfrac{\overbrace{\Sigma; \Gamma \vdash \Delta \vdash \Gamma', L_3 \longrightarrow \mathbf{L_4}}^{twig} \quad \overbrace{\Sigma; \Gamma', L_3, \mathbf{L_7} \vdash \Delta \vdash \Gamma \longrightarrow M \quad \Sigma; \Gamma', L_3, \mathbf{L_8} \vdash \Delta \vdash \Gamma \longrightarrow M}^{trunks}}{\Sigma; \Gamma', \mathbf{L_3} \vdash \Delta \vdash \Gamma \longrightarrow M} \vee\mathsf{L}^{frame}(L_3)$$

Note that the local context Γ and the accessible context Γ', L_3 in the conclusion sequent are switched in the twig sequent (because of the $\supset\mathsf{L}^{frame}$ rule application in the corresponding focused thread).

The last derived rules generated from L_1^+ are the following:

$$\dfrac{\overbrace{\Sigma; \mathbf{L_3} \vdash \Delta \vdash \Gamma, L_2 \longrightarrow M}^{trunks}}{\Sigma \vdash \Delta \vdash \Gamma, \mathbf{L_2} \longrightarrow M} \Diamond\mathsf{L}^{local}(L_2) \qquad \dfrac{\overbrace{\Sigma; \Gamma', L_2; \mathbf{L_3} \vdash \Delta \vdash \Gamma \longrightarrow M}^{trunks}}{\Sigma; \Gamma', \mathbf{L_2} \vdash \Delta \vdash \Gamma \longrightarrow M} \Diamond\mathsf{L}^{frame}(L_2)$$

where the corresponding focused threads begin with an application of $\Diamond\mathsf{L}^{local}$ or $\Diamond\mathsf{L}^{frame}$ and whose principal formula is L_2. Although we have not considered subformulas with sign $=$ or $\approx$ (since the example query formula L_1 does not contain modal connective $\Box$), they are also similarly treated.

4.2 Backward Proof Search

Given a query formula L, we perform backward proof search in a specialized inference system $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ consisting of only those derived rules generated from L and the $\mathbf{G_{IS5}}$ right rules; in other words, $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ does not use the $\mathbf{G_{IS5}}$ left rules since their applications are concealed by the derived rules. Let $R_1, \cdots, R_n$ be the inference rules of $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$. Then, the backward proof search for $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ is a simple recursive procedure as follows:

(1) Let S be the current sequent to prove. Suppose that $R_1, \cdots, R_{i-1}$ failed to prove S and R_i can be applied to S. We try R_i.
(2) If there is no premise sequent, the proof is done.
(3) Otherwise, let $S_1, \cdots, S_k$ be the premise sequents generated by the application of R_i. We recursively call the procedure for each S_j.
(3-1) If all the premises are proved, the proof is done.
(3-2) Otherwise, we backtrack to the parent sequent S to try the next inference rule.
(4) If all the rules failed to prove S, we report that it is unprovable.

To prove the query formula L, we simply apply this procedure to the sequent $\cdot \vdash \cdot \vdash \cdot \longrightarrow L$.

Figure 5 shows an example proof tree for L_1 which is constructed by backward proof search using the derived rules introduced in the previous section. For ease of reference, we also include the signed subformulas of L_1 and its derived rules in Figure 5. For backward proof search, we annotate each context with a unique identifier. For example, the global context is annotated with a subscript g and the initial local context with l. The name of each derived rule instance includes an additional context id which indicates in which context the principal formula resides. For example, the application of the rule $\Diamond\mathsf{L}^{local}(L_2, l)$ analyzes a subformula L_2 in the local context l. It introduces a new accessible context with a subscript 2 where 2 means that the context is created from the labeled subformula L_2. (Note that a new context is created only by the rule $\Diamond\mathsf{L}$ or $\Box\mathsf{R}$.) In the rightmost subderivation $\mathcal{D}_2$, the rule $\Diamond\mathsf{R}_b$ exchanges an accessible context $\{L_3, L_8\}_2$ with the local context $\{L_2, L_5\}_l$. Then, the rule $\mathsf{Id}^{local}(L_8, 2)$ uses a subformula L_8 in the local context $\{L_3, L_8\}_2$ to complete the proof.

The abovementioned recursive procedure itself is not guaranteed to terminate since the derived rules together with the right rules may be applied indefinitely many times, and thus in order to ensure its termination, we adopt a simple bookkeeping method similar to the one presented in [11]. The key idea is that if there exists a derivation of a given query sequent, then there exists another one where in each path from the query sequent to each leaf sequent, each derived rule instance is used at most once. Technically, if the same instance were used more than once, we can collapse them into one by exploiting the contraction property.

Specifically, we associate a $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ sequent with nine bookkeeping sets: $\Box\mathsf{L}^{\mathcal{R}}$, $\Diamond\mathsf{L}^{\mathcal{R}}$, $\vee\mathsf{L}^{\mathcal{R}}$, $\bot\mathsf{L}^{\mathcal{R}}$, $\mathsf{Id}^{\mathcal{R}}$, $\Box\mathsf{L}^{\mathcal{T}}$, $\Diamond\mathsf{L}^{\mathcal{T}}$, $\Box\mathsf{R}^{\mathcal{R}}$, and $\Diamond\mathsf{R}^{\mathcal{R}}$. The first five bookkeeping sets contain the labels, *i.e.*, pairs of the context ids and the labels of the principal formulas, of the corresponding derived rule instances that have been applied so far and thus should not be applied again. In other words, during the proof search, we do not try the rule instance in the bookkeeping sets. For example, in Figure 5c, after the application of the rule instance $\Diamond\mathsf{L}^{local}(L_2, l)$, we add the label (L_2, l) into the bookkeeping set $\Diamond\mathsf{L}^{\mathcal{R}}$ so that it cannot be applied again. It is unnecessary to apply it again since it only produces an already available accessible context $\{L_3\}_2$. Second, the labels contained in $\Box\mathsf{L}^{\mathcal{T}}$ and $\Diamond\mathsf{L}^{\mathcal{T}}$ indicate on which rule instance's twigs the current sequent is located. Since a twig sequent has the same antecedent as the conclusion sequent, we should not apply the same rule instance on the twig

$$(\overbrace{\overbrace{\Diamond(\overbrace{\overbrace{\Diamond \overbrace{A}^{L_5^+}}^{L_4^+} \supset \overbrace{\overbrace{B}^{L_7^-,L_7^\sim} \vee \overbrace{C}^{L_8^-,L_8^\sim}}^{L_6^\sim}}^{L_3^-,L_3^\sim})}^{L_2^-,L_2^\sim} \supset \overbrace{\overbrace{A}^{L_5^-,L_5^\sim} \supset \overbrace{\overbrace{\Diamond \overbrace{B}^{L_7^+}}^{L_{11}^+} \vee \overbrace{\Diamond \overbrace{C}^{L_8^+}}^{L_{12}^+}}^{L_{10}^+}}^{L_9^+}}^{L_1^+})^+$$

(a) The Signed Subformulas of L_1

$$\frac{}{\Sigma \vdash \Delta \vdash \Gamma, \mathbf{L_i} \longrightarrow L_i}\ \mathsf{Id}^{local}(L_i) \quad \text{where } i = 5, 7, 8$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, L_3 \longrightarrow \mathbf{L_4} \quad \Sigma \vdash \Delta \vdash \Gamma, L_3, \mathbf{L_7} \longrightarrow M \quad \Sigma \vdash \Delta \vdash \Gamma, L_3, \mathbf{L_8} \longrightarrow M}{\Sigma \vdash \Delta \vdash \Gamma, \mathbf{L_3} \longrightarrow M}\ \vee\mathsf{L}^{local}(L_3)$$

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma', L_3 \longrightarrow \mathbf{L_4} \quad \Sigma; \Gamma', L_3, \mathbf{L_7} \vdash \Delta \vdash \Gamma \longrightarrow M \quad \Sigma; \Gamma', L_3, \mathbf{L_8} \vdash \Delta \vdash \Gamma \longrightarrow M}{\Sigma; \Gamma', \mathbf{L_3} \vdash \Delta \vdash \Gamma \longrightarrow M}\ \vee\mathsf{L}^{frame}(L_3)$$

$$\frac{\Sigma; \mathbf{L_3} \vdash \Delta \vdash \Gamma, L_2 \longrightarrow M}{\Sigma \vdash \Delta \vdash \Gamma, \mathbf{L_2} \longrightarrow M}\ \Diamond\mathsf{L}^{local}(L_2) \qquad \frac{\Sigma; \Gamma', L_2; \mathbf{L_3} \vdash \Delta \vdash \Gamma \longrightarrow M}{\Sigma; \Gamma', \mathbf{L_2} \vdash \Delta \vdash \Gamma \longrightarrow M}\ \Diamond\mathsf{L}^{frame}(L_2)$$

(b) The Derived Rules for L_1

$$\cfrac{\cfrac{\cfrac{}{\{L_3\}_2 \vdash \{\}_g \vdash \{L_2, \mathbf{L_5}\}_l \longrightarrow L_5}\ \mathsf{Id}^{local}(L_5, l)}{\{L_2, L_5\}_l \vdash \{\}_g \vdash \{L_3\}_2 \longrightarrow \mathbf{L_4}}\ \Diamond\mathsf{R}_b \qquad \mathcal{D}_1 \qquad \mathcal{D}_2}{\cfrac{\cfrac{\cfrac{\{\mathbf{L_3}\}_2 \vdash \{\}_g \vdash \{L_2, L_5\}_l \longrightarrow L_9}{\{L_3\}_2 \vdash \{\}_g \vdash \{L_2\}_l \longrightarrow \mathbf{L_9}}\ \supset\mathsf{R}}{\vdash \{\}_g \vdash \{\mathbf{L_2}\}_l \longrightarrow L_9}\ \Diamond\mathsf{L}^{local}(L_2, l)}{\vdash \{\}_g \vdash \{\}_l \longrightarrow \mathbf{L_1}}\ \supset\mathsf{R}}\ \vee\mathsf{L}^{frame}(L_3, 2)$$

$$\text{where} \quad \mathcal{D}_1 = \cfrac{\cfrac{\cfrac{}{\{L_2, L_5\}_l \vdash \{\}_g \vdash \{L_3, \mathbf{L_7}\}_2 \longrightarrow L_7}\ \mathsf{Id}^{local}(L_7, 2)}{\{L_3, L_7\}_2 \vdash \{\}_g \vdash \{L_2, L_5\}_l \longrightarrow \mathbf{L_{11}}}\ \Diamond\mathsf{R}_b}{\{L_3, L_7\}_2 \vdash \{\}_g \vdash \{L_2, L_5\}_l \longrightarrow \mathbf{L_{10}}}\ \vee\mathsf{R}_l$$

$$\mathcal{D}_2 = \cfrac{\cfrac{\cfrac{}{\{L_2, L_5\}_l \vdash \{\}_g \vdash \{L_3, \mathbf{L_8}\}_2 \longrightarrow L_8}\ \mathsf{Id}^{local}(L_8, 2)}{\{L_3, L_8\}_2 \vdash \{\}_g \vdash \{L_2, L_5\}_l \longrightarrow \mathbf{L_{12}}}\ \Diamond\mathsf{R}_b}{\{L_3, L_8\}_2 \vdash \{\}_g \vdash \{L_2, L_5\}_l \longrightarrow \mathbf{L_{10}}}\ \vee\mathsf{R}_r$$

(c) Backward Proof Search for L_1 Using the Derived Rules

Fig. 5. An Example of Bidirectional Proof Search

sequent right away; otherwise, it may produce a cycle, for example, as shown below[1]:

$$\dfrac{\dfrac{\overset{\vdots}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B} \quad \overset{?}{\Sigma \vdash \Delta, A \vdash \Gamma_j \longrightarrow B}}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B}\ \Box\mathsf{L}^{local}(i,j) \quad \overset{?}{\Sigma \vdash \Delta, A \vdash \Gamma_j \longrightarrow C}}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C}\ \Box\mathsf{L}^{local}(i,j)$$

However, if the contexts are updated along the derivation rooted at that twig sequent, it is fine to apply the same rule instance again. Thus, whenever the contexts are modified, we make the rule instances in $\Box\mathsf{L}^{\mathcal{T}}$ and $\Diamond\mathsf{L}^{\mathcal{T}}$ available again. Finally, the bookkeeping sets $\Box\mathsf{R}^{\mathcal{R}}$ and $\Diamond\mathsf{R}^{\mathcal{R}}$ contain the principal formulas of the instances of the rules $\Box\mathsf{R}$ and $\Diamond\mathsf{R}_b$, respectively. $\Box\mathsf{R}$ creates a new context and $\Diamond\mathsf{R}_b$ exchanges the local context with an accessible context in the frame, so their applications make some of the derived rule instances available again. We prevent infinite applications of $\Box\mathsf{R}$ and $\Diamond\mathsf{R}_b$ in conjunction with other derived rules by exploiting the bookkeeping sets $\Box\mathsf{R}^{\mathcal{R}}$ and $\Diamond\mathsf{R}^{\mathcal{R}}$. The details of how we manage the bookkeeping sets are presented in Appendix G. With bookkeeping sets, we now state the termination theorem for our backward proof search algorithm as follows.

Theorem 4.2 (Termination of Backward Proof Search in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$).
Backward proof search in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ is guaranteed to terminate.

Proof. We define the size of a $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ sequent with bookkeeping sets and show that for each $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ inference rule, the size of the premise is always smaller than the size of the conclusion. The detailed proof is given in Theorem G.1 in Appendix G. □

Now we show that our backward proof search procedure based on derived rules and the bookkeeping method is sound and complete with respect to the proof system $\mathbf{G}_{\mathbf{IS5}}$.

Theorem 4.3 (Soundness of $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ with respect to $\mathbf{G}_{\mathbf{IS5}}$).
If $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ then $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ in $\mathbf{G}_{\mathbf{IS5}}$.

Theorem 4.4 (Completeness of $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ with respect to $\mathbf{G}_{\mathbf{IS5}}$).
If $\cdot \vdash \cdot \vdash \cdot \longrightarrow L$ in $\mathbf{G}_{\mathbf{IS5}}$ then $\cdot \vdash \cdot \vdash \cdot \longrightarrow L$ in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$, with the initial bookkeeping sets generated from the query formula L.

The equivalence between $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ and $\mathbf{G}_{\mathbf{IS5}}$ is shown indirectly using the intermediate system $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ whose definition is given in Appendix E. The soundness is easy because the derived rules in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ are also valid in $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ and the two systems share the right rules when ignoring the bookkeeping sets. In contrast, it is nontrivial to prove the completeness because a $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ derivation tree may include an invalid rule instance which violates a side condition of the bookkeeping method. We prove the completeness by defining a translation function that safely eliminates such invalid rule instances. For more details, we refer the reader to Theorems G.2 and G.3 in Appendix G.

4.3 Implementation

We have implemented an experimental prototype automated theorem prover for **IS5** in OCaml based on our bidirectional proof search procedure. Given a query formula L, it first generates a set of derived rules for L and performs backward proof search using those derived rules and the right rules as described in the previous section. Besides the bookkeeping method, we have also

[1]For simplicity, in this example, we use an arbitrary derived rule instance $\Box\mathsf{L}^{local}(i,j)$ with only one twig premise. In general, a derive rule may have no or many twig premises.

implemented several basic optimization techniques in the prototype. For example, for each sequent S encountered during the proof search, we add into a map data structure its proof, if any, or else the fact that there is no proof for S so that we can reuse the proof when encountering the same sequent again. Another example is to maintain a history of ancestor sequents of the current sequent. If the current sequent is stronger than or the same as any of the ancestors, then we abort the proof search for the current one. For example, $\cdot \vdash \cdot \vdash \cdot \longrightarrow C$ is stronger than $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$, and it is pointless to try to build a proof tree for the former while building a proof tree for the latter.

We evaluated our prototype implementation with 52 provable formulas and 14 unprovable formulas, which are available in [17], together with the source code. In most cases, the backward proof search using the derived rules is completed within 0.01 seconds on a MacBook Pro with 2.9 GHz Intel Core i5 and 16 GB of main memory, which demonstrates the effectiveness of search space reduction of our bidirectional proof search method. Even in the worst case, it only takes about 0.145 seconds to build a proof tree for the following formula:

$$((\Box D \equiv (C \supset \Diamond A)) \equiv \Diamond R) \supset (\Diamond R \equiv ((C \supset \Diamond A) \equiv \Box D))$$

where $A \equiv B$ denotes $(A \supset B) \wedge (B \supset A)$ for any A and B. The test suite contains more complex formulas and can be found in [17].

5 RELATED WORK AND CONCLUSION

In this paper, we presented a new label-free multi-context sequent calculus for the propositional fragment of intuitionistic modal logic **IS5** and its bidirectional decision procedure, taking advantage of both forward and backward proof search methods. We also formalized the cut elimination property of our sequent calculus in the Coq proof assistant and implemented an automated theorem prover based on our bidirectional decision procedure in OCaml.

Our work is inspired by the bidirectional decision procedures for intuitionistic modal logic **IS4** proposed in [10], and can be considered as another evidence showing that a bidirectional approach is both an efficient and effective proof search technique. Although our work builds on their idea of using derived rules in backward proof search, our system is significantly more complicated than theirs because the accessibility relation is symmetric in **IS5**. In order to capture the symmetry, we introduce a frame, *i.e.*, a set of accessible worlds, into the sequent structure as in [8], and the use of frames doubles the complexity of the system. Furthermore, due to the frame, we introduce context labeling during the backward proof search phase and use a revised notion of derived rule instances for the bookkeeping method. Another notable difference from [10] is that we do not distinguish subformula occurrences when constructing derived rules, so our system usually generates a less number of derived rules than their system, which often leads to faster proof search. For example, for the formula below, the decision procedure presented in [10] generates 592 derived rules, while our prover generates only 22 derived rules:

$$(((p_1 \equiv p_2) \equiv p_3) \equiv p_4) \equiv (p_4 \equiv (p_3 \equiv (p_2 \equiv p_1)))$$

Our sequent calculus $\mathbf{G_{IS5}}$ extends with a global context the one for **IS5** proposed in [8] and with a frame the one for **IS4** proposed in [10]. While the frame is necessary to capture the symmetry of the accessibility relation, the global context may not, that is, it would be also possible to build a bidirectional decision method on the system proposed in [8]. In this work, the reason for using a global context is twofold. First, the left rules for modality $\Box$ are presented in an uniform way, independently of the frame, thus making it simpler to deal with $\Box$. Second, it makes it easier to adopt a bidirectional decision method proposed in [10]. Indeed, most of the techniques developed in [10] are reused in this work; we mainly had to extend them to deal with the frame.

An interesting direction for future work is to develop an uniform framework for bidirectional decision methods for all the logics in the intuitionistic S5 cube, which slightly differ from each other. For example, McLaughlin and Pfenning [15] study a general focused inverse method theorem prover for the intuitionistic modal logics **K**, **D**, **K4**, **D4**, **T**, **S4**, and **S5**. In contrast to our work, they use a sequent calculus with explicit world annotations, and thus have to explicitly analyze the accessibility relation between worlds, represented as a world graph, during the proof search. They reformulate the world-graph accessibility problem into the world-path equivalence problem, which is determined by unification. Then, the proof search methods for the various logics are obtained by using different versions of the unification algorithm. Another example is the modular focused proof systems proposed by Chaudhuri *et al.*, which use label-free nested sequents [4]. They represent various modal logics by extending the base proof system NIK with additional inference rules corresponding to modal axioms such as **d**, **t**, **b**, **4**, and **5**. Then, they develop sound and complete focused proof systems which can be used as the basis for automated proof search. As future work, it would be interesting to investigate whether the bidirectional approach can be generalized for those modular proof systems such as in [4, 15].

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A SEQUENT CALCULUS $\mathbf{G_{IS5}}$ WITH EXPLICIT PROOF SIZES

Figure 6 shows inference rules for $\mathbf{G_{IS5}}$ using a sequent $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ where a proof size n is explicitly annotated with the judgment $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$. Below we show some of inversion and related properties which are needed to prove the cut elimination theorem for $\mathbf{G_{IS5}}$ and other theorems. For more details, we refer the reader to the Coq formalization available in [17].

Proposition A.1 (Inversion for $\wedge$).

(1) ($\wedge\mathsf{L}^{local}$) *If* $n \simeq \Sigma \vdash \Delta \vdash \Gamma, A \wedge B \longrightarrow C$ *then* $n \geq \Sigma \vdash \Delta \vdash \Gamma, A, B \longrightarrow C$.
(2) ($\wedge\mathsf{L}^{global}$) *If* $n \simeq \Sigma \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow C$ *then* $n \geq \Sigma \vdash \Delta, A, B \vdash \Gamma \longrightarrow C$.
(3) ($\wedge\mathsf{L}^{frame}$) *If* $n \simeq \Sigma; \Gamma', A \wedge B \vdash \Delta \vdash \Gamma \longrightarrow C$ *then* $n \geq \Sigma; \Gamma', A, B \vdash \Delta \vdash \Gamma \longrightarrow C$.
(4) ($\wedge\mathsf{R}$) *If* $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \wedge B$ *then* $n \geq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A$ *and* $n \geq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow B$.

Proof. (1) and (3) are proved by simultaneous induction on n. (2) and (4) are proved by induction on n. □

Proposition A.2 (Inversion for $\vee$).

- ($\vee\mathsf{L}^{local}$) *If* $n \simeq \Sigma \vdash \Delta \vdash \Gamma, A \vee B \longrightarrow C$ *then* $n \geq \Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow C$ *and* $n \geq \Sigma \vdash \Delta \vdash \Gamma, B \longrightarrow C$.
- ($\vee\mathsf{L}^{frame}$) *If* $n \simeq \Sigma; \Gamma', A \vee B \vdash \Delta \vdash \Gamma \longrightarrow C$ *then* $n \geq \Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C$ *and* $n \geq \Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C$.

Proof. By simultaneous induction on n. □

Proposition A.3 (Inversion for $\supset$).

- ($\supset\mathsf{R}$) *If* $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \supset B$ *then* $n \geq \Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow B$.

Proof. By induction on n. □

Proposition A.4 (Inversion for $\Box$).

(1) ($\Box\mathsf{L}^{local}$) *If* $n \simeq \Sigma \vdash \Delta \vdash \Gamma, \Box A \longrightarrow C$ *then* $n \geq \Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C$.
(2) ($\Box\mathsf{L}^{global}$) *If* $n \simeq \Sigma \vdash \Delta, \Box A \vdash \Gamma \longrightarrow C$ *then* $n \geq \Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C$.
(3) ($\Box\mathsf{L}^{frame}$) *If* $n \simeq \Sigma; \Gamma', \Box A \vdash \Delta \vdash \Gamma \longrightarrow C$ *then* $n \geq \Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow C$.
(4) ($\Box\mathsf{R}$) *If* $n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow \Box A$ *then* $n \geq \Sigma; \Gamma \vdash \Delta \vdash \cdot \longrightarrow A$.

Proof. (1) and (3) are proved by simultaneous induction on n. (2) and (4) are proved by induction on n. □

Proposition A.5 (Inversion for $\Diamond$).

(1) ($\Diamond\mathsf{L}^{local}$) *If* $n \simeq \Sigma \vdash \Delta \vdash \Gamma, \Diamond A \longrightarrow C$ *then* $n \geq \Sigma; A \vdash \Delta \vdash \Gamma \longrightarrow C$.
(2) ($\Diamond\mathsf{L}^{global}$) *If* $n \simeq \Sigma \vdash \Delta, \Diamond A \vdash \Gamma \longrightarrow C$ *then* $n \geq \Sigma; A \vdash \Delta \vdash \Gamma \longrightarrow C$.
(3) ($\Diamond\mathsf{L}^{frame}$) *If* $n \simeq \Sigma; \Gamma', \Diamond A \vdash \Delta \vdash \Gamma \longrightarrow C$ *then* $n \geq \Sigma; \Gamma'; A \vdash \Delta \vdash \Gamma \longrightarrow C$.

Proof. (1) and (3) are proved by simultaneous induction on n, and (2) by induction on n. □

Proposition A.6 (Auxiliary property for $\supset$).

- *If* $n \simeq \Sigma \vdash \Delta \vdash \Gamma, A \supset B \longrightarrow C$ *then* $n \geq \Sigma \vdash \Delta \vdash \Gamma, B \longrightarrow C$.
- *If* $n \simeq \Sigma; \Gamma', A \supset B \vdash \Delta \vdash \Gamma \longrightarrow C$ *then* $n \geq \Sigma; \Gamma', B \vdash \Delta \vdash \Gamma \longrightarrow C$.

Proof. By simultaneous induction on n. Note that B is stronger than $A \supset B$. □

$$\frac{}{0 \simeq \Sigma \vdash \Delta \vdash \Gamma, p \longrightarrow p}\;\mathsf{Id}^{local} \qquad \frac{}{0 \simeq \Sigma \vdash \Delta, p \vdash \Gamma \longrightarrow p}\;\mathsf{Id}^{global} \qquad \frac{}{0 \simeq \Sigma \vdash \Delta \vdash \Gamma, \bot \longrightarrow C}\;\bot\mathsf{L}^{local}$$

$$\frac{}{0 \simeq \Sigma \vdash \Delta, \bot \vdash \Gamma \longrightarrow C}\;\bot\mathsf{L}^{global} \qquad \frac{}{0 \simeq \Sigma; \Gamma', \bot \vdash \Delta \vdash \Gamma \longrightarrow C}\;\bot\mathsf{L}^{frame}$$

$$\frac{n \simeq \Sigma \vdash \Delta \vdash \Gamma, A, B \longrightarrow C}{n+1 \simeq \Sigma \vdash \Delta \vdash \Gamma, A \wedge B \longrightarrow C}\;\wedge\mathsf{L}^{local} \qquad \frac{n \simeq \Sigma \vdash \Delta, A, B \vdash \Gamma \longrightarrow C}{n+1 \simeq \Sigma \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow C}\;\wedge\mathsf{L}^{global}$$

$$\frac{n \simeq \Sigma; \Gamma', A, B \vdash \Delta \vdash \Gamma \longrightarrow C}{n+1 \simeq \Sigma; \Gamma', A \wedge B \vdash \Delta \vdash \Gamma \longrightarrow C}\;\wedge\mathsf{L}^{frame} \qquad \frac{n_1 \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \qquad n_2 \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow B}{n_1 + n_2 + 1 \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \wedge B}\;\wedge\mathsf{R}$$

$$\frac{n_1 \simeq \Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow C \qquad n_2 \simeq \Sigma \vdash \Delta \vdash \Gamma, B \longrightarrow C}{n_1 + n_2 + 1 \simeq \Sigma \vdash \Delta \vdash \Gamma, A \vee B \longrightarrow C}\;\vee\mathsf{L}^{local}$$

$$\frac{n_1 \simeq \Sigma \vdash \Delta, A \vee B \vdash \Gamma, A \longrightarrow C \qquad n_2 \simeq \Sigma \vdash \Delta, A \vee B \vdash \Gamma, B \longrightarrow C}{n_1 + n_2 + 1 \simeq \Sigma \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow C}\;\vee\mathsf{L}^{global}_a$$

$$\frac{n_1 \simeq \Sigma; \Gamma', A \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow C \qquad n_2 \simeq \Sigma; \Gamma', B \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow C}{n_1 + n_2 + 1 \simeq \Sigma; \Gamma' \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow C}\;\vee\mathsf{L}^{global}_b$$

$$\frac{n_1 \simeq \Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C \qquad n_2 \simeq \Sigma; \Gamma', B \vdash \Delta \vdash \Gamma \longrightarrow C}{n_1 + n_2 + 1 \simeq \Sigma; \Gamma', A \vee B \vdash \Delta \vdash \Gamma \longrightarrow C}\;\vee\mathsf{L}^{frame}$$

$$\frac{n_1 \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A}{n+1 \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \vee B}\;\vee\mathsf{R}_l \qquad \frac{n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow B}{n+1 \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \vee B}\;\vee\mathsf{R}_r$$

$$\frac{n_1 \simeq \Sigma \vdash \Delta \vdash \Gamma, A \supset B \longrightarrow A \qquad n_2 \simeq \Sigma \vdash \Delta \vdash \Gamma, B \longrightarrow C}{n_1 + n_2 + 1 \simeq \Sigma \vdash \Delta \vdash \Gamma, A \supset B \longrightarrow C}\;\supset\mathsf{L}^{local}$$

$$\frac{n_1 \simeq \Sigma \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow A \qquad n_2 \simeq \Sigma \vdash \Delta, A \supset B \vdash \Gamma, B \longrightarrow C}{n_1 + n_2 + 1 \simeq \Sigma \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow C}\;\supset\mathsf{L}^{global}_a$$

$$\frac{n_1 \simeq \Sigma; \Gamma \vdash \Delta, A \supset B \vdash \Gamma' \longrightarrow A \qquad n_2 \simeq \Sigma; \Gamma', B \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow C}{n_1 + n_2 + 1 \simeq \Sigma; \Gamma' \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow C}\;\supset\mathsf{L}^{global}_b$$

$$\frac{n_1 \simeq \Sigma; \Gamma \vdash \Delta \vdash \Gamma', A \supset B \longrightarrow A \qquad n_2 \simeq \Sigma; \Gamma', B \vdash \Delta \vdash \Gamma \longrightarrow C}{n_1 + n_2 + 1 \simeq \Sigma; \Gamma', A \supset B \vdash \Delta \vdash \Gamma \longrightarrow C}\;\supset\mathsf{L}^{frame}$$

$$\frac{n \simeq \Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow B}{n+1 \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \supset B}\;\supset\mathsf{R} \qquad \frac{n \simeq \Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C}{n+1 \simeq \Sigma \vdash \Delta \vdash \Gamma, \Box A \longrightarrow C}\;\Box\mathsf{L}^{local}$$

$$\frac{n \simeq \Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C}{n+1 \simeq \Sigma \vdash \Delta, \Box A \vdash \Gamma \longrightarrow C}\;\Box\mathsf{L}^{global} \qquad \frac{n \simeq \Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow C}{n+1 \simeq \Sigma; \Gamma', \Box A \vdash \Delta \vdash \Gamma \longrightarrow C}\;\Box\mathsf{L}^{frame}$$

$$\frac{n \simeq \Sigma; \Gamma \vdash \Delta \vdash \cdot \longrightarrow A}{n+1 \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow \Box A}\;\Box\mathsf{R} \qquad \frac{n \simeq \Sigma; A \vdash \Delta \vdash \Gamma \longrightarrow C}{n+1 \simeq \Sigma \vdash \Delta \vdash \Gamma, \Diamond A \longrightarrow C}\;\Diamond\mathsf{L}^{local}$$

$$\frac{n \simeq \Sigma; A \vdash \Delta \vdash \Gamma \longrightarrow C}{n+1 \simeq \Sigma \vdash \Delta, \Diamond A \vdash \Gamma \longrightarrow C}\;\Diamond\mathsf{L}^{global} \qquad \frac{n \simeq \Sigma; \Gamma'; A \vdash \Delta \vdash \Gamma \longrightarrow C}{n+1 \simeq \Sigma; \Gamma', \Diamond A \vdash \Delta \vdash \Gamma \longrightarrow C}\;\Diamond\mathsf{L}^{frame}$$

$$\frac{n \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow A}{n+1 \simeq \Sigma \vdash \Delta \vdash \Gamma \longrightarrow \Diamond A}\;\Diamond\mathsf{R}_a \qquad \frac{n \simeq \Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow A}{n+1 \simeq \Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow \Diamond A}\;\Diamond\mathsf{R}_b$$

Fig. 6. Sequent Calculus $\mathbf{G_{IS5}}$ with Explicit Proof Sizes

$$\frac{}{\Sigma \vdash \Gamma, p \vdash p}\ [Id](p \in \text{Prop}) \qquad \frac{}{\Sigma \vdash \Gamma, \bot \vdash C}\ [\bot^1] \qquad \frac{}{\Sigma; \Gamma', \bot \vdash \Gamma \vdash C}\ [\bot^2]$$

$$\frac{\Sigma \vdash \Gamma, A, B \vdash C}{\Sigma \vdash \Gamma, A \wedge B \vdash C}\ [\wedge_L] \qquad \frac{\Sigma; \Gamma', A, B \vdash \Gamma \vdash C}{\Sigma; \Gamma', A \wedge B \vdash \Gamma \vdash C}\ [\wedge_{LL}] \qquad \frac{\Sigma \vdash \Gamma \vdash A \qquad \Sigma \vdash \Gamma \vdash B}{\Sigma \vdash \Gamma \vdash A \wedge B}\ [\wedge_R]$$

$$\frac{\Sigma \vdash \Gamma, A \vdash C \qquad \Sigma \vdash \Gamma, B \vdash C}{\Sigma \vdash \Gamma, A \vee B \vdash C}\ [\vee_L] \qquad \frac{\Sigma; \Gamma', A \vdash \Gamma \vdash C \qquad \Sigma; \Gamma', B \vdash \Gamma \vdash C}{\Sigma; \Gamma', A \vee B \vdash \Gamma \vdash C}\ [\vee_{LL}]$$

$$\frac{\Sigma \vdash \Gamma \vdash A}{\Sigma \vdash \Gamma \vdash A \vee B}\ [\vee_R^1] \qquad \frac{\Sigma \vdash \Gamma \vdash B}{\Sigma \vdash \Gamma \vdash A \vee B}\ [\vee_R^2] \qquad \frac{\Sigma \vdash \Gamma, A \supset B \vdash A \qquad \Sigma \vdash \Gamma, B \vdash C}{\Sigma \vdash \Gamma, A \supset B \vdash C}\ [\supset_L]$$

$$\frac{\Sigma; \Gamma \vdash \Gamma', A \supset B \vdash A \qquad \Sigma; \Gamma', B \vdash \Gamma \vdash C}{\Sigma; \Gamma', A \supset B \vdash \Gamma \vdash C}\ [\supset_{LL}] \qquad \frac{\Sigma \vdash \Gamma, A \vdash B}{\Sigma \vdash \Gamma \vdash A \supset B}\ [\supset_R]$$

$$\frac{\Sigma \vdash \Gamma, \Box A, A \vdash C}{\Sigma \vdash \Gamma, \Box A \vdash C}\ [\Box_L^1] \qquad \frac{\Sigma; \Gamma', A \vdash \Gamma, \Box A \vdash C}{\Sigma; \Gamma' \vdash \Gamma, \Box A \vdash C}\ [\Box_L^2] \qquad \frac{\Sigma; \Gamma', \Box A \vdash \Gamma, A \vdash C}{\Sigma; \Gamma', \Box A \vdash \Gamma \vdash C}\ [\Box_{LL}^1]$$

$$\frac{\Sigma; \Gamma', \Box A, A \vdash \Gamma \vdash C}{\Sigma; \Gamma', \Box A \vdash \Gamma \vdash C}\ [\Box_{LL}^{2a}] \qquad \frac{\Sigma; \Gamma'', A; \Gamma', \Box A \vdash \Gamma \vdash C}{\Sigma; \Gamma''; \Gamma', \Box A \vdash \Gamma \vdash C}\ [\Box_{LL}^{2b}] \qquad \frac{\Sigma; \Gamma \vdash\ \vdash A}{\Sigma \vdash \Gamma \vdash \Box A}\ [\Box_R]$$

$$\frac{\Sigma; A \vdash \Gamma \vdash C}{\Sigma \vdash \Gamma, \Diamond A \vdash C}\ [\Diamond_L] \qquad \frac{\Sigma; A; \Gamma' \vdash \Gamma \vdash C}{\Sigma; \Gamma', \Diamond A \vdash \Gamma \vdash C}\ [\Diamond_{LL}] \qquad \frac{\Sigma \vdash \Gamma \vdash A}{\Sigma \vdash \Gamma \vdash \Diamond A}\ [\Diamond_R^1] \qquad \frac{\Sigma; \Gamma \vdash \Gamma' \vdash A}{\Sigma; \Gamma' \vdash \Gamma \vdash \Diamond A}\ [\Diamond_R^2]$$

Fig. 7. Sequent Calculus **G** [8]

B SOUNDNESS AND COMPLETENESS OF $\mathbf{G_{IS5}}$

Figure 7 shows the cut-free sequent calculus $\mathbf{G}$[2] for intuitionistic modal logic **IS5** proposed by Galmiche and Salhi [8], which is sound and complete with respect to the Kripke semantics of **IS5**. The main difference with our calculus $\mathbf{G_{IS5}}$ is that while their sequent exploits only a frame and a local context, we also introduce a global context containing universally valid formulas.[3] The correctness of $\mathbf{G_{IS5}}$ with respect to the Kripke semantics of **IS5** is established by showing that it is sound and complete with respect to **G**.

Theorem B.1 (Completeness of $\mathbf{G_{IS5}}$). *If* $\Sigma \vdash \Gamma \vdash C$ *then* $\Sigma \vdash \cdot \vdash \Gamma \longrightarrow C$.

Proof. By induction on a derivation of $\Sigma \vdash \Gamma \vdash C$. The proof is quite straightforward since the inference rules in **G** except for the left rules for $\Box$ are also included in $\mathbf{G_{IS5}}$, by simply replacing a sequent of the form $\Sigma \vdash \Gamma \vdash C$ in **G** with $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$. If the final rule used in the derivation of $\Sigma \vdash \Gamma \vdash C$ is one of $[\Box_L^1]$, $[\Box_L^2]$, $[\Box_{LL}^1]$, $[\Box_{LL}^{2a}]$, $[\Box_{LL}^{2b}]$, and $[\Box_R]$, we prove the case by using the inversion property of $\Box$ and the contraction property of $\mathbf{G_{IS5}}$. □

Theorem B.2 (Soundess of $\mathbf{G_{IS5}}$). *If* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ *then* $\Sigma \vdash \Box\Delta, \Gamma \vdash C$ *where* $\Box\Delta$ *is defined as* $\Box A_1, \Box A_2, .., \Box A_n$ *if* $\Delta = A_1, A_2, .., A_n$.

Proof. By induction on a derivation of $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$. Since global contexts are not used in **G**, the soundness proof involves a translation of a global context Δ into a set of valid formulas in **G**, which we denote $\Box\Delta$. Moreover, whereas the completeness of $\mathbf{G_{IS5}}$ is established with respect to

[2]The sequent calculus proposed in [8] is named $\mathbf{G_{IS5}}$. As we use the same name, we call their sequent calculus simply **G** to avoid confusion.

[3]In [8], a frame is called *LL-context* and denoted by a symbol G instead of Σ.

the cut-free **G**, to simplify the proof, the soundness is established with respect to **G** with cut rules as follows:

$$\frac{\Sigma \vdash \Gamma \vdash A \qquad \Sigma \vdash \Gamma, A \vdash C}{\Sigma \vdash \Gamma \vdash C}\ [Cut^1] \qquad\qquad \frac{\Sigma; \Gamma \vdash \Gamma' \vdash A \qquad \Sigma; \Gamma', A \vdash \Gamma \vdash C}{\Sigma; \Gamma' \vdash \Gamma \vdash C}\ [Cut^2]$$

In **G**, both $[Cut^1]$ and $[Cut^2]$ are admissible.

The proof is mostly completed by applying induction hypotheses followed by the inference rule of **G** corresponding to the final rule used in the derivation of $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$. If the final rule analyzes Δ, the proof also exploits the structural properties of **G** and Lemma B.3 stated below, and is completed by applying the appropriate rule for $\Box$. ◻

Lemma B.3.

(1) If $\Sigma; \Gamma' \vdash \Gamma, \Box\Delta \vdash C$ *then* $\Sigma; \Gamma', \Box\Delta \vdash \Gamma \vdash C$.
(2) If $\Sigma; \Gamma', \Box\Delta \vdash \Gamma \vdash C$ *then* $\Sigma; \Gamma' \vdash \Gamma, \Box\Delta \vdash C$.

Proof. By induction on a derivation of $\Sigma; \Gamma' \vdash \Gamma, \Box\Delta \vdash C$ and $\Sigma; \Gamma', \Box\Delta \vdash \Gamma \vdash C$. ◻

Since the accessibility relation in **IS5** is an equivalence relation, the same $\Box A$ in every accessible world has the same implications regarding the provability of the same formula C. Lemma B.3 generalizes this fact to an arbitrary number of necessity formulas in the system **G**, *i.e.* $\Box\Delta$.

C SIGNED $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$

Figures 8 and 9 show a signed version of $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$, which are simply derived from the inference rules in Figures 2 and 3 based on the definition of signed subformulas. $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ and its signed version are equivalent in terms of the provability, as stated below.

Theorem C.1. $\Sigma \vdash \Delta \vdash \Gamma \longmapsto C$ *if and only if* $\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+$.

Proof. ($\Longrightarrow$) Given a derivation $\mathcal{D}$ of $\Sigma \vdash \Delta \vdash \Gamma \longmapsto C$, we can uniquely annotate the sign to each subformula used in $\mathcal{D}$, and thus obtain a derivation of $\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+$. ($\Longleftarrow$) Given a derivation $\mathcal{E}$ of $\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+$, we easily obtain a derivation of $\Sigma \vdash \Delta \vdash \Gamma \longmapsto C$ by simply erasing the signs of subformulas in $\mathcal{E}$. ◻

D EQUIVALENCE OF $\mathbf{G}_{\mathbf{IS5}}$ AND $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$

We show the equivalence of $\mathbf{G}_{\mathbf{IS5}}$ and $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$ through two intermediate equivalent systems. The first intermediate system $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ is shown in Figures 10 and 11, and differs from $\mathbf{G}_{\mathbf{IS5}}$ in two aspects. First, in $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$, the principal formula in the conclusion of every left rule remains in the premises and may be re-analyzed. In the proof of Theorem D.1, given a derivation $\mathcal{D}$ of $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$, this property allows us to exploit inversion properties such as Proposition A.2 on the sequent obtained by applying the induction hypothesis on the premises of the final rule in $\mathcal{D}$. Second, the left rules for $\wedge$ ($\wedge\mathsf{L}^{local}$, $\wedge\mathsf{L}^{global}_a$, $\wedge\mathsf{L}^{global}_b$, $\wedge\mathsf{L}^{frame}$) select only one of A_1 and A_2 from the principal formula $A_1 \wedge A_2$ and use it in the premise sequent, which is also the case for $\mathbf{G}^{\mathbf{F}}_{\mathbf{IS5}}$.

Theorem D.1 (Soundness of $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ with respect to $\mathbf{G}_{\mathbf{IS5}}$).
If $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$.

Proof. By induction on a derivation of $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$. We give some interesting cases; the other cases are similarly proved.

Case $\dfrac{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B, A \longrightarrow_I C}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B \longrightarrow_I C}\ \wedge\mathsf{L}^{local}_l$

$$\frac{}{\cdot \vdash \cdot \vdash \cdot \rhd p^{\sim} \longmapsto p^{+}}\ \mathsf{Id}^{local} \qquad \frac{}{\cdot \vdash \cdot \vdash \cdot \rhd \rhd p^{\approx} \longmapsto p^{+}}\ \mathsf{Id}^{global} \qquad \frac{}{\cdot \vdash \cdot \vdash \cdot \rhd \bot^{\sim} \longmapsto C^{+}}\ \bot\mathsf{L}^{local}$$

$$\frac{}{\cdot \vdash \cdot \vdash \cdot \rhd \rhd \bot^{\approx} \longmapsto C^{+}}\ \bot\mathsf{L}_a^{global} \qquad \frac{}{\cdot \rhd \rhd \bot^{\approx} \vdash \cdot \vdash \cdot \longmapsto C^{+}}\ \bot\mathsf{L}_b^{global} \qquad \frac{}{\cdot \rhd \bot^{\sim} \vdash \cdot \vdash \cdot \longmapsto C^{+}}\ \bot\mathsf{L}^{frame}$$

$$\frac{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-} \rhd A^{\sim} \longmapsto C^{+}}{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-}, A^{-} \longmapsto C^{+}}\ \mathsf{ch}^{local} \qquad \frac{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-} \rhd \rhd A^{\approx} \longmapsto C^{+}}{\Sigma^{-} \vdash \Delta^{=}, A^{=} \vdash \Gamma^{-} \longmapsto C^{+}}\ \mathsf{ch}_a^{global}$$

$$\frac{\Sigma^{-}; \Gamma_1^{-} \rhd \rhd A^{\approx} \vdash \Delta^{=} \vdash \Gamma_2^{-} \longmapsto C^{+}}{\Sigma^{-}; \Gamma_1^{-} \vdash \Delta^{=}, A^{=} \vdash \Gamma_2^{-} \longmapsto C^{+}}\ \mathsf{ch}_b^{global} \qquad \frac{\Sigma^{-}; \Gamma_1^{-} \rhd A^{\sim} \vdash \Delta^{=} \vdash \Gamma_2^{-} \longmapsto C^{+}}{\Sigma^{-}; \Gamma_1^{-}, A^{-} \vdash \Delta^{=} \vdash \Gamma_2^{-} \longmapsto C^{+}}\ \mathsf{ch}^{frame}$$

$$\frac{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-} \rhd A_i^{\sim} \longmapsto C^{+}}{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-} \rhd (A_1 \wedge A_2)^{\sim} \longmapsto C^{+}}\ \wedge\mathsf{L}^{local} \qquad \frac{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-} \rhd \rhd A_i^{\approx} \longmapsto C^{+}}{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-} \rhd \rhd (A_1 \wedge A_2)^{\approx} \longmapsto C^{+}}\ \wedge\mathsf{L}_a^{global}$$

$$\frac{\Sigma^{-}; \Gamma_1^{-} \rhd \rhd A_i^{\approx} \vdash \Delta^{=} \vdash \Gamma_2^{-} \longmapsto C^{+}}{\Sigma^{-}; \Gamma_1^{-} \rhd \rhd (A_1 \wedge A_2)^{\approx} \vdash \Delta^{=} \vdash \Gamma_2^{-} \longmapsto C^{+}}\ \wedge\mathsf{L}_b^{global}$$

$$\frac{\Sigma^{-}; \Gamma_1^{-} \rhd A_i^{\sim} \vdash \Delta^{=} \vdash \Gamma_2^{-} \longmapsto C^{+}}{\Sigma^{-}; \Gamma_1^{-} \rhd (A_1 \wedge A_2)^{\sim} \vdash \Delta^{=} \vdash \Gamma_2^{-} \longmapsto C^{+}}\ \wedge\mathsf{L}^{frame}$$

$$\frac{\Sigma_1^{-} \vdash \Delta_1^{=} \vdash \Gamma_1^{-} \longmapsto A_1^{+} \qquad \Sigma_2^{-} \vdash \Delta_2^{=} \vdash \Gamma_2^{-} \longmapsto A_2^{+}}{\Sigma_1^{-}; \Sigma_2^{-} \vdash \Delta_1^{=}, \Delta_2^{=} \vdash \Gamma_1^{-}, \Gamma_2^{-} \longmapsto (A_1 \wedge A_2)^{+}}\ \wedge\mathsf{R}$$

$$\frac{\Sigma_1^{-} \vdash \Delta_1^{=} \vdash \Gamma_1^{-}, A_1^{-} \longmapsto C^{+} \qquad \Sigma_2^{-} \vdash \Delta_2^{=} \vdash \Gamma_2^{-}, A_2^{-} \longmapsto C^{+}}{\Sigma_1^{-}; \Sigma_2^{-} \vdash \Delta_1^{=}, \Delta_2^{=} \vdash \Gamma_1^{-}, \Gamma_2^{-} \rhd (A_1 \vee A_2)^{\sim} \longmapsto C^{+}}\ \vee\mathsf{L}^{local}$$

$$\frac{\Sigma_1^{-} \vdash \Delta_1^{=} \vdash \Gamma_1^{-}, A_1^{-} \longmapsto C^{+} \qquad \Sigma_2^{-} \vdash \Delta_2^{=} \vdash \Gamma_2^{-}, A_2^{-} \longmapsto C^{+}}{\Sigma_1^{-}; \Sigma_2^{-} \vdash \Delta_1^{=}, \Delta_2^{=} \vdash \Gamma_1^{-}, \Gamma_2^{-} \rhd \rhd (A_1 \vee A_2)^{\approx} \longmapsto C^{+}}\ \vee\mathsf{L}_a^{global}$$

$$\frac{\Sigma_1^{-}; \Gamma_1^{-}, A_1^{-} \vdash \Delta_1^{=} \vdash \Gamma_3^{-} \longmapsto C^{+} \qquad \Sigma_2^{-}; \Gamma_2^{-}, A_2^{-} \vdash \Delta_2^{=} \vdash \Gamma_4^{-} \longmapsto C^{+}}{\Sigma_1^{-}; \Sigma_2^{-}; \Gamma_1^{-}, \Gamma_2^{-} \rhd \rhd (A_1 \vee A_2)^{\approx} \vdash \Delta_1^{=}, \Delta_2^{=} \vdash \Gamma_3^{-}, \Gamma_4^{-} \longmapsto C^{+}}\ \vee\mathsf{L}_b^{global}$$

$$\frac{\Sigma_1^{-}; \Gamma_1^{-}, A_1^{-} \vdash \Delta_1^{=} \vdash \Gamma_3^{-} \longmapsto C^{+} \qquad \Sigma_2^{-}; \Gamma_2^{-}, A_2^{-} \vdash \Delta_2^{=} \vdash \Gamma_4^{-} \longmapsto C^{+}}{\Sigma_1^{-}; \Sigma_2^{-}; \Gamma_1^{-}, \Gamma_2^{-} \rhd (A_1 \vee A_2)^{\sim} \vdash \Delta_1^{=}, \Delta_2^{=} \vdash \Gamma_3^{-}, \Gamma_4^{-} \longmapsto C^{+}}\ \vee\mathsf{L}^{frame}$$

$$\frac{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-} \longmapsto A_1^{+}}{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-} \longmapsto (A_1 \vee A_2)^{+}}\ \vee\mathsf{R}_l \qquad \frac{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-} \longmapsto A_2^{+}}{\Sigma^{-} \vdash \Delta^{=} \vdash \Gamma^{-} \longmapsto (A_1 \vee A_2)^{+}}\ \vee\mathsf{R}_r$$

$$i \in \{1, 2\}$$

Fig. 8. Signed $\mathbf{G}^{\mathsf{F}}_{\mathsf{IS5}}$

(1) $\Sigma \vdash \Delta \vdash \Gamma, A \wedge B, A \longrightarrow C$	by I.H.
(2) $\Sigma \vdash \Delta \vdash \Gamma, A, B, A \longrightarrow C$	by Proposition A.1 (inversion on $\wedge\mathsf{L}^{local}$)
(3) $\Sigma \vdash \Delta \vdash \Gamma, A, B \longrightarrow C$	by Proposition 2.4 (contraction on A)
(4) $\Sigma \vdash \Delta \vdash \Gamma, A \wedge B \longrightarrow C$	by $\wedge\mathsf{L}^{local}$ with (3)

$$\frac{\Sigma_1^- \vdash \Delta_1^= \vdash \Gamma_1^- \longmapsto A_1^+ \qquad \Sigma_2^- \vdash \Delta_2^= \vdash \Gamma_2^- \rhd A_2^{\sim} \longmapsto C^+}{\Sigma_1^-; \Sigma_2^- \vdash \Delta_1^=, \Delta_2^= \vdash \Gamma_1^-, \Gamma_2^- \rhd (A_1 \supset A_2)^{\sim} \longmapsto C^+} \supset\mathsf{L}^{local}$$

$$\frac{\Sigma_1^- \vdash \Delta_1^= \vdash \Gamma_1^- \longmapsto A_1^+ \qquad \Sigma_2^- \vdash \Delta_2^= \vdash \Gamma_2^- \rhd \rhd A_2^{\approx} \longmapsto C^+}{\Sigma_1^-; \Sigma_2^- \vdash \Delta_1^=, \Delta_2^= \vdash \Gamma_1^-, \Gamma_2^- \rhd \rhd (A_1 \supset A_2)^{\approx} \longmapsto C^+} \supset\mathsf{L}_a^{global}$$

$$\frac{\Sigma_1^-; \Gamma_3^- \vdash \Delta_1^= \vdash \Gamma_1^- \longmapsto A_1^+ \qquad \Sigma_2^-; \Gamma_2^- \rhd \rhd A_2^{\approx} \vdash \Delta_2^= \vdash \Gamma_4^- \longmapsto C^+}{\Sigma_1^-; \Sigma_2^-; \Gamma_1^-, \Gamma_2^- \rhd \rhd (A_1 \supset A_2)^{\approx} \vdash \Delta_1^=, \Delta_2^= \vdash \Gamma_3^-, \Gamma_4^- \longmapsto C^+} \supset\mathsf{L}_b^{global}$$

$$\frac{\Sigma_1^-; \Gamma_3^- \vdash \Delta_1^= \vdash \Gamma_1^- \longmapsto A_1^+ \qquad \Sigma_2^-; \Gamma_2^- \rhd A_2^{\sim} \vdash \Delta_2^= \vdash \Gamma_4^- \longmapsto C^+}{\Sigma_1^-; \Sigma_2^-; \Gamma_1^-, \Gamma_2^- \rhd (A_1 \supset A_2)^{\sim} \vdash \Delta_1^=, \Delta_2^= \vdash \Gamma_3^-, \Gamma_4^- \longmapsto C^+} \supset\mathsf{L}^{frame}$$

$$\frac{\Sigma^- \vdash \Delta^= \vdash \Gamma^-, A_1^- \longmapsto A_2^+}{\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto (A_1 \supset A_2)^+} \supset\mathsf{R}_a \qquad \frac{\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto A_2^+}{\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto (A_1 \supset A_2)^+} \supset\mathsf{R}_b$$

$$\frac{\Sigma^- \vdash \Delta^=, A^= \vdash \Gamma^- \longmapsto C^+}{\Sigma^- \vdash \Delta^= \vdash \Gamma^- \rhd (\Box A)^{\sim} \longmapsto C^+} \Box\mathsf{L}^{local} \qquad \frac{\Sigma^- \vdash \Delta^=, A^= \vdash \Gamma^- \longmapsto C^+}{\Sigma^- \vdash \Delta^= \vdash \Gamma^- \rhd \rhd (\Box A)^{\approx} \longmapsto C^+} \Box\mathsf{L}_a^{global}$$

$$\frac{\Sigma^-; \Gamma_1^- \vdash \Delta^=, A^= \vdash \Gamma_2^- \longmapsto C^+}{\Sigma^-; \Gamma_1^- \rhd \rhd (\Box A)^{\approx} \vdash \Delta^= \vdash \Gamma_2^- \longmapsto C^+} \Box\mathsf{L}_b^{global} \qquad \frac{\Sigma^-; \Gamma_1^- \vdash \Delta^=, A^= \vdash \Gamma_2^- \longmapsto C^+}{\Sigma^-; \Gamma_1^- \rhd (\Box A)^{\sim} \vdash \Delta^= \vdash \Gamma_2^- \longmapsto C^+} \Box\mathsf{L}^{frame}$$

$$\frac{\Sigma^-; \Gamma^- \vdash \Delta^= \vdash \cdot \longmapsto A^+}{\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto (\Box A)^+} \Box\mathsf{R} \qquad \frac{\Sigma^-; A^- \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+}{\Sigma^- \vdash \Delta^= \vdash \Gamma^- \rhd (\Diamond A)^{\sim} \longmapsto C^+} \Diamond\mathsf{L}^{local}$$

$$\frac{\Sigma^-; A^- \vdash \Delta^= \vdash \Gamma^- \longmapsto C^+}{\Sigma^- \vdash \Delta^= \vdash \Gamma^- \rhd \rhd (\Diamond A)^{\approx} \longmapsto C^+} \Diamond\mathsf{L}_a^{global} \qquad \frac{\Sigma^-; \Gamma_1^-; A^- \vdash \Delta^= \vdash \Gamma_2^- \longmapsto C^+}{\Sigma^-; \Gamma_1^- \rhd \rhd (\Diamond A)^{\approx} \vdash \Delta^= \vdash \Gamma_2^- \longmapsto C^+} \Diamond\mathsf{L}_b^{global}$$

$$\frac{\Sigma^-; \Gamma_1^-; A^- \vdash \Delta^= \vdash \Gamma_2^- \longmapsto C^+}{\Sigma^-; \Gamma_1^- \rhd (\Diamond A)^{\sim} \vdash \Delta^= \vdash \Gamma_2^- \longmapsto C^+} \Diamond\mathsf{L}^{frame} \qquad \frac{\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto A^+}{\Sigma^- \vdash \Delta^= \vdash \Gamma^- \longmapsto (\Diamond A)^+} \Diamond\mathsf{R}_a$$

$$\frac{\Sigma^-; \Gamma_2^- \vdash \Delta^= \vdash \Gamma_1^- \longmapsto A^+}{\Sigma^-; \Gamma_1^- \vdash \Delta^= \vdash \Gamma_2^- \longmapsto (\Diamond A)^+} \Diamond\mathsf{R}_b$$

Fig. 9. Signed $\mathbf{G}_{\mathsf{IS5}}^{\mathsf{F}}$ (Continued)

Case $\dfrac{\Sigma \vdash \Delta, A \wedge B \vdash \Gamma, A \longrightarrow_I C}{\Sigma \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow_I C} \wedge\mathsf{L}_a^{global}$

(1) $\Sigma \vdash \Delta, A \wedge B \vdash \Gamma, A \longrightarrow C$ by I.H.
(2) $\Sigma \vdash \Delta, A, B \vdash \Gamma, A \longrightarrow C$ by Proposition A.1 (inversion on $\wedge\mathsf{L}^{global}$)
(3) $\Sigma \vdash \Delta, A, B \vdash \Gamma \longrightarrow C$ by Proposition 2.4 (contraction on A)
(4) $\Sigma \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow C$ by $\wedge\mathsf{L}^{global}$ with (3)

Case $\dfrac{\mathcal{D} :: \Sigma; \Gamma', A \vee B, A \vdash \Delta \vdash \Gamma \longrightarrow_I C \qquad \mathcal{E} :: \Sigma; \Gamma', A \vee B, B \vdash \Delta \vdash \Gamma \longrightarrow_I C}{\Sigma; \Gamma', A \vee B \vdash \Delta \vdash \Gamma \longrightarrow_I C} \vee\mathsf{L}^{frame}$

(1) $\Sigma; \Gamma', A \vee B, A \vdash \Delta \vdash \Gamma \longrightarrow C$ by I.H. on $\mathcal{D}$

$$\frac{}{\Sigma \vdash \Delta \vdash \Gamma, p \longrightarrow_I p}\ \mathsf{Id}^{local} \qquad \frac{}{\Sigma \vdash \Delta, p \vdash \Gamma \longrightarrow_I p}\ \mathsf{Id}^{global} \qquad \frac{}{\Sigma \vdash \Delta \vdash \Gamma, \bot \longrightarrow_I C}\ \bot\mathsf{L}^{local}$$

$$\frac{}{\Sigma \vdash \Delta, \bot \vdash \Gamma \longrightarrow_I C}\ \bot\mathsf{L}^{global} \qquad \frac{}{\Sigma; \Gamma', \bot \vdash \Delta \vdash \Gamma \longrightarrow_I C}\ \bot\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A_1 \wedge A_2, A_i \longrightarrow_I C}{\Sigma \vdash \Delta \vdash \Gamma, A_1 \wedge A_2 \longrightarrow_I C}\ \wedge\mathsf{L}^{local} \qquad \frac{\Sigma \vdash \Delta, A_1 \wedge A_2 \vdash \Gamma, A_i \longrightarrow_I C}{\Sigma \vdash \Delta, A_1 \wedge A_2 \vdash \Gamma \longrightarrow_I C}\ \wedge\mathsf{L}_a^{global}$$

$$\frac{\Sigma; \Gamma', A_i \vdash \Delta, A_1 \wedge A_2 \vdash \Gamma \longrightarrow_I C}{\Sigma; \Gamma' \vdash \Delta, A_1 \wedge A_2 \vdash \Gamma \longrightarrow_I C}\ \wedge\mathsf{L}_b^{global} \qquad \frac{\Sigma; \Gamma', A_1 \wedge A_2, A_i \vdash \Delta \vdash \Gamma \longrightarrow_I C}{\Sigma; \Gamma', A_1 \wedge A_2 \vdash \Delta \vdash \Gamma \longrightarrow_I C}\ \wedge\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I A_1 \qquad \Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I A_2}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I A_1 \wedge A_2}\ \wedge\mathsf{R}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A \vee B, A \longrightarrow_I C \qquad \Sigma \vdash \Delta \vdash \Gamma, A \vee B, B \longrightarrow_I C}{\Sigma \vdash \Delta \vdash \Gamma, A \vee B \longrightarrow_I C}\ \vee\mathsf{L}^{local}$$

$$\frac{\Sigma \vdash \Delta, A \vee B \vdash \Gamma, A \longrightarrow_I C \qquad \Sigma \vdash \Delta, A \vee B \vdash \Gamma, B \longrightarrow_I C}{\Sigma \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow_I C}\ \vee\mathsf{L}_a^{global}$$

$$\frac{\Sigma; \Gamma', A \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow_I C \qquad \Sigma; \Gamma', B \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow_I C}{\Sigma; \Gamma' \vdash \Delta, A \vee B \vdash \Gamma \longrightarrow_I C}\ \vee\mathsf{L}_b^{global}$$

$$\frac{\Sigma; \Gamma', A \vee B, A \vdash \Delta \vdash \Gamma \longrightarrow_I C \qquad \Sigma; \Gamma', A \vee B, B \vdash \Delta \vdash \Gamma \longrightarrow_I C}{\Sigma; \Gamma', A \vee B \vdash \Delta \vdash \Gamma \longrightarrow_I C}\ \vee\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I A \vee B}\ \vee\mathsf{R}_l \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I B}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I A \vee B}\ \vee\mathsf{R}_r$$

$$i \in \{1, 2\}$$

Fig. 10. Intermediate Sequent Calculus $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$

(2) $\Sigma; \Gamma', A, A \vdash \Delta \vdash \Gamma \longrightarrow C$ — by Proposition A.2 (inversion on $\vee\mathsf{L}^{frame}$)
(3) $\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow C$ — by Proposition 2.4 (contraction on A)
(4) $\Sigma; \Gamma', B \vdash \Delta \vdash \Gamma \longrightarrow C$ — by I.H. on $\mathcal{E}$, Proposition A.2, and Proposition 2.4
(5) $\Sigma; \Gamma', A \vee B \vdash \Delta \vdash \Gamma \longrightarrow C$ — by $\vee\mathsf{L}^{frame}$ with (3) and (4)

Case $\dfrac{\Sigma \vdash \Delta, A \vdash \Gamma, \Box A \longrightarrow_I C}{\Sigma \vdash \Delta \vdash \Gamma, \Box A \longrightarrow_I C}\ \Box\mathsf{L}^{local}$

(1) $\Sigma \vdash \Delta, A \vdash \Gamma, \Box A \longrightarrow C$ — by I.H.
(2) $\Sigma \vdash \Delta, A, A \vdash \Gamma \longrightarrow C$ — by Proposition A.4 (inversion on $\Box\mathsf{L}^{local}$)
(3) $\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C$ — by Proposition 2.4 (contraction on A)
(4) $\Sigma \vdash \Delta \vdash \Gamma, \Box A \longrightarrow C$ — by $\Box\mathsf{L}^{local}$ with (3)

□

The following lemmas are needed to prove the completeness of $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ with respect to $\mathbf{G}_{\mathbf{IS5}}$ and other theorems.

Lemma D.2 (Weakening).

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A \supset B \longrightarrow_I A \qquad \Sigma \vdash \Delta \vdash \Gamma, A \supset B, B \longrightarrow_I C}{\Sigma \vdash \Delta \vdash \Gamma, A \supset B \longrightarrow_I C} \supset \mathsf{L}^{local}$$

$$\frac{\Sigma \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow_I A \qquad \Sigma \vdash \Delta, A \supset B \vdash \Gamma, B \longrightarrow_I C}{\Sigma \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow_I C} \supset \mathsf{L}_a^{global}$$

$$\frac{\Sigma; \Gamma \vdash \Delta, A \supset B \vdash \Gamma' \longrightarrow_I A \qquad \Sigma; \Gamma', B \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow_I C}{\Sigma; \Gamma' \vdash \Delta, A \supset B \vdash \Gamma \longrightarrow_I C} \supset \mathsf{L}_b^{global}$$

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma', A \supset B \longrightarrow_I A \qquad \Sigma; \Gamma', A \supset B, B \vdash \Delta \vdash \Gamma \longrightarrow_I C}{\Sigma; \Gamma', A \supset B \vdash \Delta \vdash \Gamma \longrightarrow_I C} \supset \mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow_I B}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I A \supset B} \supset \mathsf{R} \qquad \frac{\Sigma \vdash \Delta, A \vdash \Gamma, \Box A \longrightarrow_I C}{\Sigma \vdash \Delta \vdash \Gamma, \Box A \longrightarrow_I C} \Box\mathsf{L}^{local}$$

$$\frac{\Sigma \vdash \Delta, \Box A, A \vdash \Gamma \longrightarrow_I C}{\Sigma \vdash \Delta, \Box A \vdash \Gamma \longrightarrow_I C} \Box\mathsf{L}^{global} \qquad \frac{\Sigma; \Gamma', \Box A \vdash \Delta, A \vdash \Gamma \longrightarrow_I C}{\Sigma; \Gamma', \Box A \vdash \Delta \vdash \Gamma \longrightarrow_I C} \Box\mathsf{L}^{frame}$$

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \cdot \longrightarrow_I A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I \Box A} \Box\mathsf{R} \qquad \frac{\Sigma; A \vdash \Delta \vdash \Gamma, \Diamond A \longrightarrow_I C}{\Sigma \vdash \Delta \vdash \Gamma, \Diamond A \longrightarrow_I C} \Diamond\mathsf{L}^{local}$$

$$\frac{\Sigma; A \vdash \Delta, \Diamond A \vdash \Gamma \longrightarrow_I C}{\Sigma \vdash \Delta, \Diamond A \vdash \Gamma \longrightarrow_I C} \Diamond\mathsf{L}^{global} \qquad \frac{\Sigma; \Gamma', \Diamond A; A \vdash \Delta \vdash \Gamma \longrightarrow_I C}{\Sigma; \Gamma', \Diamond A \vdash \Delta \vdash \Gamma \longrightarrow_I C} \Diamond\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I \Diamond A} \Diamond\mathsf{R}_a \qquad \frac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow_I A}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow_I \Diamond A} \Diamond\mathsf{R}_b$$

Fig. 11. Intermediate Sequent Calculus $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ (Continued)

(1) If $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow_I C$.
(2) If $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow_I C$.
(3) If $\Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow_I C$.
(4) If $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma; \Gamma \vdash \Delta \vdash \Gamma \longrightarrow_I C$.

Proof. By simultaneous induction on the structure of the given derivation. As in Proposition 2.2, the weakening property for $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ is also size-preserving. □

Lemma D.3 (Contraction).

(1) If $\Sigma \vdash \Delta \vdash \Gamma, A, A \longrightarrow_I C$ *then* $\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow_I C$.
(2) If $\Sigma \vdash \Delta, A, A \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow_I C$.
(3) If $\Sigma; \Gamma', A, A \vdash \Delta \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow_I C$.
(4) If $\Sigma \vdash \Delta, A \vdash \Gamma, A \longrightarrow_I C$ *then* $\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow_I C$.
(5) If $\Sigma; \Gamma', A \vdash \Delta, A \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow_I C$.
(6) If $\Sigma; \Gamma \vdash \Delta \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$.
(7) If $\Sigma; \Gamma', \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow_I C$.

Proof. By simultaneous induction on the structure of the given derivation. As in Proposition 2.4, the contraction property for $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ is also size-preserving. □

Lemma D.4. *If* $\Sigma \vdash \Delta, A, B \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow_I C$.

Proof. By induction on a derivation of $\Sigma \vdash \Delta, A, B \vdash \Gamma \longrightarrow_I C$, using Lemma D.2 where necessary. □

We now show the completeness of $\mathbf{G^I_{IS5}}$.

Theorem D.5 (Completeness of $\mathbf{G^I_{IS5}}$ with respect to $\mathbf{G_{IS5}}$). *If* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ *then* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$.

Proof. We define a recursive function f that takes a $\mathbf{G_{IS5}}$ derivation of $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ and returns a corresponding $\mathbf{G^I_{IS5}}$ derivation of $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$. The termination of f is guaranteed because a recursive call on f always takes a $\mathbf{G_{IS5}}$ derivation of a smaller size. We give some interesting cases; the remaining cases are similarly defined.

$$f\left(\cfrac{\overset{\mathcal{E}}{\Sigma \vdash \Delta \vdash \Gamma, A, B \longrightarrow C}}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B \longrightarrow C}\ \wedge\mathsf{L}^{local}\right) \overset{\text{def}}{=} \cfrac{\cfrac{f\left(\overset{\mathcal{E}'}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B, A, B \longrightarrow C}\right)}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B, A \longrightarrow_I C}\ \wedge\mathsf{L}^{local}_r}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B \longrightarrow_I C}\ \wedge\mathsf{L}^{local}_l$$

We obtain $\mathcal{E}' :: \Sigma \vdash \Delta \vdash \Gamma, A \wedge B, A, B \longrightarrow C$ by the size-preserving weakening property (Proposition 2.2) with $\mathcal{E} :: \Sigma \vdash \Delta \vdash \Gamma, A, B \longrightarrow C$. Since $\mathcal{E}'$ is smaller than or equal to $\mathcal{E}$, we may recursively call f on $\mathcal{E}'$.

$$f\left(\cfrac{\overset{\mathcal{E}}{\Sigma \vdash \Delta, A, B \vdash \Gamma \longrightarrow C}}{\Sigma \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow C}\ \wedge\mathsf{L}^{global}\right) \overset{\text{def}}{=} \cfrac{f\left(\overset{\mathcal{E}}{\Sigma \vdash \Delta, A, B \vdash \Gamma \longrightarrow C}\right)}{\Sigma \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow_I C}\ \text{By Lemma D.4}$$

$$f\left(\cfrac{\overset{\mathcal{E}}{\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C}}{\Sigma \vdash \Delta, \Box A \vdash \Gamma \longrightarrow C}\ \Box\mathsf{L}^{global}\right) \overset{\text{def}}{=} \cfrac{f\left(\overset{\mathcal{E}'}{\Sigma \vdash \Delta, \Box A, A \vdash \Gamma \longrightarrow C}\right)}{\Sigma \vdash \Delta, \Box A \vdash \Gamma \longrightarrow_I C}\ \Box\mathsf{L}^{global}$$

We obtain $\mathcal{E}' :: \Sigma \vdash \Delta, \Box A, A \vdash \Gamma \longrightarrow C$ by the size-preserving weakening property (Proposition 2.2) with $\mathcal{E} :: \Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow C$ where $|\mathcal{E}'| \leq |\mathcal{E}|$. □

E EQUIVALENCE OF $\mathbf{G^I_{IS5}}$ AND $\mathbf{G^K_{IS5}}$

The second intermediate system $\mathbf{G^K_{IS5}}$ is shown in Figures 12 and 13, and we show that it is equivalent to $\mathbf{G^I_{IS5}}$, thus establishing the equivalence of $\mathbf{G^K_{IS5}}$ and $\mathbf{G_{IS5}}$. $\mathbf{G^K_{IS5}}$ differs from $\mathbf{G^I_{IS5}}$ in that it has a focus on the principal formula in the left rules and axioms, and includes ch rules that move the stoup formula into one of the contexts as in $\mathbf{G^F_{IS5}}$. The ch rules of $\mathbf{G^K_{IS5}}$ are, however, different from those of $\mathbf{G^F_{IS5}}$ in that when reading the rules in a bottom up way, the principal formula in the conclusion sequent remains in the premise sequent. That is, the principal formula is duplicated into the stoup in the premise. Therefore, one important invariant of $\mathbf{G^K_{IS5}}$ is that the stoup formula in every focused sequent of $\mathbf{G^K_{IS5}}$ left rules is a subformula of some formula that occur in one of the contexts of the sequent. Below we first prove the soundness of $\mathbf{G^K_{IS5}}$ with respect to $\mathbf{G^I_{IS5}}$.

Theorem E.1 (Soundness of $\mathbf{G^K_{IS5}}$ with respect to $\mathbf{G^I_{IS5}}$).

(1) If $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K C$ *then* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$.
(2) If $\Sigma \vdash \Delta \vdash \Gamma \rhd A \longrightarrow_K C$ *then* $\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow_I C$.
(3) If $\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd A \longrightarrow_K C$ *then* $\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow_I C$.
(4) If $\Sigma; \Gamma' \rhd \rhd A \vdash \Delta \vdash \Gamma \longrightarrow_K C$ *then* $\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow_I C$.
(5) If $\Sigma; \Gamma' \rhd A \vdash \Delta \vdash \Gamma \longrightarrow_K C$ *then* $\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow_I C$.

Proof. By simultaneous induction on the structure of the given derivation. We give some interesting cases; the other cases are similarly proved.

$$\frac{}{\Sigma \vdash \Delta \vdash \Gamma \rhd p \longrightarrow_K p}\ \mathsf{Id}^{local} \qquad \frac{}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd p \longrightarrow_K p}\ \mathsf{Id}^{global}$$

$$\frac{}{\Sigma \vdash \Delta \vdash \Gamma \rhd \bot \longrightarrow_K C}\ \bot\mathsf{L}^{local} \qquad \frac{}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd \bot \longrightarrow_K C}\ \bot\mathsf{L}_a^{global}$$

$$\frac{}{\Sigma; \Gamma' \rhd \rhd \bot \vdash \Delta \vdash \Gamma \longrightarrow_K C}\ \bot\mathsf{L}_b^{global} \qquad \frac{}{\Sigma; \Gamma' \rhd \bot \vdash \Delta \vdash \Gamma \longrightarrow_K C}\ \bot\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A \rhd A \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow_K C}\ \mathsf{ch}^{local} \qquad \frac{\Sigma \vdash \Delta, A \vdash \Gamma \rhd \rhd A \longrightarrow_K C}{\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow_K C}\ \mathsf{ch}_a^{global}$$

$$\frac{\Sigma; \Gamma' \rhd \rhd A \vdash \Delta, A \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow_K C}\ \mathsf{ch}_b^{global} \qquad \frac{\Sigma; \Gamma', A \rhd A \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow_K C}\ \mathsf{ch}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \rhd A_i \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd A_1 \wedge A_2 \longrightarrow_K C}\ \wedge\mathsf{L}^{local} \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd A_i \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd A_1 \wedge A_2 \longrightarrow_K C}\ \wedge\mathsf{L}_a^{global}$$

$$\frac{\Sigma; \Gamma' \rhd \rhd A_i \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd \rhd A_1 \wedge A_2 \vdash \Delta \vdash \Gamma \longrightarrow_K C}\ \wedge\mathsf{L}_b^{global} \qquad \frac{\Sigma; \Gamma' \rhd A_i \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd A_1 \wedge A_2 \vdash \Delta \vdash \Gamma \longrightarrow_K C}\ \wedge\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A_1 \qquad \Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A_2}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A_1 \wedge A_2}\ \wedge\mathsf{R}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow_K C \qquad \Sigma \vdash \Delta \vdash \Gamma, B \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd A \vee B \longrightarrow_K C}\ \vee\mathsf{L}^{local}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow_K C \qquad \Sigma \vdash \Delta \vdash \Gamma, B \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd A \vee B \longrightarrow_K C}\ \vee\mathsf{L}_a^{global}$$

$$\frac{\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow_K C \qquad \Sigma; \Gamma', B \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd \rhd A \vee B \vdash \Delta \vdash \Gamma \longrightarrow_K C}\ \vee\mathsf{L}_b^{global}$$

$$\frac{\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow_K C \qquad \Sigma; \Gamma', B \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd A \vee B \vdash \Delta \vdash \Gamma \longrightarrow_K C}\ \vee\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A \vee B}\ \vee\mathsf{R}_l \qquad \frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K B}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A \vee B}\ \vee\mathsf{R}_r$$

$$i \in \{1, 2\}$$

Fig. 12. Intermediate Sequent Calculus $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ with a Focus

Case $\dfrac{\Sigma \vdash \Delta, A \vdash \Gamma \rhd \rhd A \longrightarrow_K C}{\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow_K C}\ \mathsf{ch}_a^{global}$

(1) $\Sigma \vdash \Delta, A \vdash \Gamma, A \longrightarrow_I C$ by I.H.
(2) $\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow_I C$ by Lemma D.3 (contraction on A)

Case $\dfrac{\Sigma \vdash \Delta \vdash \Gamma \rhd A_i \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd A_1 \wedge A_2 \longrightarrow_K C}\ \wedge\mathsf{L}^{local}$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A \qquad \Sigma \vdash \Delta \vdash \Gamma \rhd B \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd A \supset B \longrightarrow_K C} \supset\mathsf{L}^{local}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A \qquad \Sigma \vdash \Delta \vdash \Gamma \rhd \rhd B \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd A \supset B \longrightarrow_K C} \supset\mathsf{L}_a^{global}$$

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow_K A \qquad \Sigma; \Gamma' \rhd \rhd B \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd \rhd A \supset B \vdash \Delta \vdash \Gamma \longrightarrow_K C} \supset\mathsf{L}_b^{global}$$

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow_K A \qquad \Sigma; \Gamma' \rhd B \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd A \supset B \vdash \Delta \vdash \Gamma \longrightarrow_K C} \supset\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow_K B}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A \supset B} \supset\mathsf{R} \qquad \frac{\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \Box A \longrightarrow_K C} \Box\mathsf{L}^{local}$$

$$\frac{\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd \Box A \longrightarrow_K C} \Box\mathsf{L}_a^{global} \qquad \frac{\Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd \rhd \Box A \vdash \Delta \vdash \Gamma \longrightarrow_K C} \Box\mathsf{L}_b^{global}$$

$$\frac{\Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd \Box A \vdash \Delta \vdash \Gamma \longrightarrow_K C} \Box\mathsf{L}^{frame} \qquad \frac{\Sigma; \Gamma \vdash \Delta \vdash \cdot \longrightarrow_K A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K \Box A} \Box\mathsf{R}$$

$$\frac{\Sigma; A \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \Diamond A \longrightarrow_K C} \Diamond\mathsf{L}^{local} \qquad \frac{\Sigma; A \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd \Diamond A \longrightarrow_K C} \Diamond\mathsf{L}_a^{global}$$

$$\frac{\Sigma; \Gamma'; A \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd \rhd \Diamond A \vdash \Delta \vdash \Gamma \longrightarrow_K C} \Diamond\mathsf{L}_b^{global} \qquad \frac{\Sigma; \Gamma'; A \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd \Diamond A \vdash \Delta \vdash \Gamma \longrightarrow_K C} \Diamond\mathsf{L}^{frame}$$

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K \Diamond A} \Diamond\mathsf{R}_a \qquad \frac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow_K A}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow_K \Diamond A} \Diamond\mathsf{R}_b$$

Fig. 13. Intermediate Sequent Calculus $\mathbf{G}^{\mathsf{K}}_{\mathsf{IS5}}$ with a Focus (Continued)

(1) $\Sigma \vdash \Delta \vdash \Gamma, A_i \longrightarrow_I C$ — by I.H.
(2) $\Sigma \vdash \Delta \vdash \Gamma, A_1 \wedge A_2, A_i \longrightarrow_I C$ — by Lemma D.2 (weakening)
(3) $\Sigma \vdash \Delta \vdash \Gamma, A_1 \wedge A_2 \longrightarrow_I C$ — by $\wedge\mathsf{L}^{local}$

Case $\dfrac{\Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd \Box A \vdash \Delta \vdash \Gamma \longrightarrow_K C} \Box\mathsf{L}^{frame}$

(1) $\Sigma; \Gamma' \vdash \Delta, A \vdash \Gamma \longrightarrow_I C$ — by I.H.
(2) $\Sigma; \Gamma', \Box A \vdash \Delta, A \vdash \Gamma \longrightarrow_I C$ — by Lemma D.2 (weakening)
(3) $\Sigma; \Gamma', \Box A \vdash \Delta \vdash \Gamma \longrightarrow_I C$ — by $\Box\mathsf{L}^{frame}$

Case $\dfrac{\Sigma; A \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd \Diamond A \longrightarrow_K C} \Diamond\mathsf{L}_a^{global}$

(1) $\Sigma; A \vdash \Delta \vdash \Gamma \longrightarrow_I C$ — by I.H.
(2) $\Sigma; A \vdash \Delta \vdash \Gamma, \Diamond A \longrightarrow_I C$ — by Lemma D.2 (weakening)
(3) $\Sigma \vdash \Delta \vdash \Gamma, \Diamond A \longrightarrow_I C$ — by $\Diamond\mathsf{L}^{local}$

□

In order to prove the completeness of $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ with respect to $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$, we first need to show that if a sequent is provable in $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$, there always exists a derivation tree of a certain shape, *i.e.,* that can be transformed into a $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ derivation tree. The following lemma allows us to reorder left-rule applications in a $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ derivation tree.

Lemma E.2. *Assume that* N_1 *is a* $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ *left rule and* N_2 *does not analyze the formula introduced by* N_1.

If $\dfrac{\dfrac{\overset{\mathcal{D}}{\Sigma_3 \vdash \Delta_3 \vdash \Gamma_3 \longrightarrow_I C_3}}{\Sigma_2 \vdash \Delta_2 \vdash \Gamma_2 \longrightarrow_I C_2}\,\mathsf{N}_2}{\Sigma_1 \vdash \Delta_1 \vdash \Gamma_1 \longrightarrow_I C_1}\,\mathsf{N}_1$ *then* $\dfrac{\dfrac{\overset{\mathcal{D}}{\Sigma_3 \vdash \Delta_3 \vdash \Gamma_3 \longrightarrow_I C_3}}{\Sigma_2' \vdash \Delta_2' \vdash \Gamma_2' \longrightarrow_I C_2'}\,\mathsf{N}_1'}{\Sigma_1 \vdash \Delta_1 \vdash \Gamma_1 \longrightarrow_I C_1}\,\mathsf{N}_2$ *where* N_1' *is simply* N_1 *or its dual rule—for any connective* $\otimes$*, we say that* $\otimes\mathsf{L}^{frame}$ *is a dual of* $\otimes\mathsf{L}^{local}$ *and vice versa.*

Proof. By case analysis on the rules N_1 and N_2. For example, the following proof tree:

$$\dfrac{\overset{\mathcal{D}_1}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma, A_1 \supset A_2 \longrightarrow_I A_1} \quad \dfrac{\overset{\mathcal{D}_2}{\Sigma; \Gamma, A_1 \supset A_2, A_2 \vdash \Delta \vdash \Gamma' \longrightarrow_I C}}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma, A_1 \supset A_2, A_2 \longrightarrow_I \Diamond C}\,\Diamond\mathsf{R}_b}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma, A_1 \supset A_2 \longrightarrow_I \Diamond C}\,\supset\mathsf{L}^{local}$$

is transformed into the following:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma, A_1 \supset A_2 \longrightarrow_I A_1} \quad \overset{\mathcal{D}_2}{\Sigma; \Gamma, A_1 \supset A_2, A_2 \vdash \Delta \vdash \Gamma' \longrightarrow_I C}}{\Sigma; \Gamma, A_1 \supset A_2 \vdash \Delta \vdash \Gamma' \longrightarrow_I C}\,\supset\mathsf{L}^{frame}}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma, A_1 \supset A_2 \longrightarrow_I \Diamond C}\,\Diamond\mathsf{R}_b$$

As another example, the following proof tree:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Sigma; \Gamma, A \vee B, A \vdash \Delta \vdash \cdot \longrightarrow_I C}}{\Sigma \vdash \Delta \vdash \Gamma, A \vee B, A \longrightarrow_I \Box C}\,\Box\mathsf{R} \quad \dfrac{\overset{\mathcal{D}_2}{\Sigma; \Gamma, A \vee B, B \vdash \Delta \vdash \cdot \longrightarrow_I C}}{\Sigma \vdash \Delta \vdash \Gamma, A \vee B, B \longrightarrow_I \Box C}\,\Box\mathsf{R}}{\Sigma \vdash \Delta \vdash \Gamma, A \vee B \longrightarrow_I \Box C}\,\vee\mathsf{L}^{local}$$

is transformed into the following:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Sigma; \Gamma, A \vee B, A \vdash \Delta \vdash \cdot \longrightarrow_I C} \quad \overset{\mathcal{D}_2}{\Sigma; \Gamma, A \vee B, B \vdash \Delta \vdash \cdot \longrightarrow_I C}}{\Sigma; \Gamma, A \vee B \vdash \Delta \vdash \cdot \longrightarrow_I C}\,\vee\mathsf{L}^{local}}{\Sigma \vdash \Delta \vdash \Gamma, A \vee B \longrightarrow_I \Box C}\,\Box\mathsf{R}$$

The other cases are similarly proved. □

Moreover, the following properties hold for $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$.

- Given a derivation $\mathcal{D} :: \Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$, we may eliminate any left-rule application in $\mathcal{D}$ if the subformula introduced by the left rule is never used in $\mathcal{D}$. Such left-rule applications are redundant. The proof tree obtained from $\mathcal{D}$ by removing redundant rule applications and the subformulas introduced by those rules is still valid for $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$.
- If $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$ is provable, then there exists a proof tree in which each formula in each context is analyzed by a left rule at most once. This is mainly due to the contraction property of $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$.

Thanks to Lemma E.2 and the abovementioned properties, given a $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ derivation $\mathcal{D}$ for a sequent $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$, we can consider another proof tree $\mathcal{E}$ (1) where every left-rule application analyzes the subformula introduced by the parent rule application if the parent is also a left rule, (2) which has no redundant left-rule applications, and (3) where each formula in each context is analyzed at most once. The first condition implies that if there is a $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ proof tree, then there also exists a focused proof tree. In the completeness proof below, we consider a proof tree that satisfies the above three conditions.

Theorem E.3 (Completeness of $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ with respect to $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$). *If* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I C$ *then* $\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K C$.

Proof. We give a recursive function f that takes a $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ derivation that satisfies the abovementioned three conditions and returns a corresponding $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ derivation. Since the inference rules of $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ and $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ are mostly in one-to-one correspondence, the function f simply replaces a $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ rule application with the corresponding $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ rule application. In doing so, there are two subtle points. First, since a $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ proof has a focus on the principal formula of each left rule application, f needs to insert a ch rule into each boundary of left- and right-rule applications, as illustrated by the following example.

$$f\left(\cfrac{\cfrac{\overset{\mathcal{D}}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B, A \longrightarrow_I C}}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B \longrightarrow_I C}\ \wedge\mathsf{L}^{local}}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_I A \wedge B \supset C}\ \supset\mathsf{R}\right) = \cfrac{\cfrac{\cfrac{\overset{f(\mathcal{D})}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B \rhd A \longrightarrow_K C}}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B \rhd A \wedge B \longrightarrow_K C}\ \wedge\mathsf{L}^{local}}{\Sigma \vdash \Delta \vdash \Gamma, A \wedge B \longrightarrow_K C}\ \mathsf{ch}^{local}}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K A \wedge B \supset C}\ \supset\mathsf{R}$$

In order to track the boundaries between left- and right-rule applications, we introduce another function g that is mutually recursive with f. Only f inserts ch rules during the translation. The translation above also illustrates the second subtle point. Whereas in the $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ proof in the left side, the subformulas introduced by analyzing the principal formula $A \wedge B$ are added into the local context, it is not the case in the $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ proof in the right side. For example, in the derivation $f(\mathcal{D}) :: \Sigma \vdash \Delta \vdash \Gamma, A \wedge B \rhd A \longrightarrow_K C$ above, the focused formula A is not copied into the local context Γ. Therefore, we should remove the subformulas introduced by left-rule applications in the $\mathbf{G}^{\mathbf{I}}_{\mathbf{IS5}}$ proof during the translation into a $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ proof.

Below we give some interesting cases of the definitions of f and g where difference operations between contexts are defined as usual. One thing to note is that g takes a focused formula as another argument and is defined only on the axioms and some of the left rules.

Case:

$$f\left(\cfrac{}{\Sigma \vdash \Delta \vdash \Gamma, p \longrightarrow_I p}\ \mathsf{Id}^{local} \mid \Sigma' \mid \Gamma'\right) = \cfrac{\cfrac{}{\Sigma - \Sigma' \vdash \Delta \vdash \Gamma, p - \Gamma' \rhd p \longrightarrow_K p}\ \mathsf{Id}^{local}}{\Sigma - \Sigma' \vdash \Delta \vdash \Gamma, p - \Gamma' \longrightarrow_K p}\ \mathsf{ch}^{local}$$

Case:

$$g\left(\cfrac{}{\Sigma \vdash \Delta \vdash \Gamma, p \longrightarrow_I p}\ \mathsf{Id}^{local} \mid \Sigma' \mid \Gamma' \rhd p\right) = \cfrac{}{\Sigma - \Sigma' \vdash \Delta \vdash \Gamma, p - \Gamma' \rhd p \longrightarrow_K p}\ \mathsf{Id}^{local}$$

Case:

$$f\left(\cfrac{\overset{\mathcal{D}}{\Sigma \vdash \Delta, \Box A, A \vdash \Gamma \longrightarrow_I C}}{\Sigma \vdash \Delta, \Box A \vdash \Gamma \longrightarrow_I C}\ \Box\mathsf{L}^{global} \mid \Sigma' \mid \Gamma'\right) = \cfrac{\cfrac{f\left(\overset{\mathcal{D}}{\Sigma \vdash \Delta, \Box A, A \vdash \Gamma \longrightarrow_I C} \mid \Sigma' \mid \Gamma'\right)}{\Sigma - \Sigma' \vdash \Delta, \Box A \vdash \Gamma - \Gamma' \rhd \rhd \Box A \longrightarrow_K C}\ \Box\mathsf{L}^{global}_a}{\Sigma - \Sigma' \vdash \Delta, \Box A \vdash \Gamma - \Gamma' \longrightarrow_K C}\ \mathsf{ch}^{global}_a$$

Case:

$$f\left(\cfrac{\overset{\mathcal{D}}{\Sigma; \Gamma', A \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow_I C}}{\Sigma; \Gamma' \vdash \Delta, A \wedge B \vdash \Gamma \longrightarrow_I C}\ \wedge\mathsf{L}^{global}_b \mid \Sigma'; \Gamma'' \mid \Gamma'''\right) =$$

$$\cfrac{\cfrac{g\left(\begin{array}{c}\mathcal{D}\\ \Sigma;\Gamma',A\vdash\Delta,A\wedge B\vdash\Gamma\longrightarrow_I C \mid \Sigma';\Gamma'',A\rhd\rhd A\mid\Gamma'''\end{array}\right)}{\Sigma;\Gamma'-\Sigma';\Gamma''\rhd\rhd A\wedge B\vdash\Delta,A\wedge B\vdash\Gamma-\Gamma'''\longrightarrow_K C}\wedge\mathsf{L}_b^{global}}{\Sigma;\Gamma'-\Sigma';\Gamma''\vdash\Delta,A\wedge B\vdash\Gamma-\Gamma'''\longrightarrow_K C}\mathsf{ch}_b^{global}$$

Case:

$$f\left(\cfrac{\begin{array}{c}\mathcal{D}\\ \Sigma;\Gamma\vdash\Delta\vdash\cdot\longrightarrow_I A\end{array}}{\Sigma\vdash\Delta\vdash\Gamma\longrightarrow_I \Box A}\Box\mathsf{R} \mid \Sigma'\mid\Gamma'\right)=\cfrac{f\left(\begin{array}{c}\mathcal{D}\\ \Sigma;\Gamma\vdash\Delta\vdash\cdot\longrightarrow_I A\mid\Sigma';\Gamma'\mid\cdot\end{array}\right)}{\Sigma-\Sigma'\vdash\Delta\vdash\Gamma-\Gamma'\longrightarrow_K \Box A}\Box\mathsf{R}$$

The remaining cases are similarly defined. With f and g defined, given a $\mathbf{G^I_{IS5}}$ derivation $\mathcal{D} :: \Sigma\vdash\Delta\vdash\Gamma\longrightarrow_I C$ where $|\Sigma| = n$, $f(\mathcal{D} :: \Sigma\vdash\Delta\vdash\Gamma\longrightarrow_I C \mid \cdot_1;\ldots;\cdot_n \mid \cdot)$ gives a $\mathbf{G^K_{IS5}}$ derivation. □

F EQUIVALENCE OF $\mathbf{G^K_{IS5}}$ AND $\mathbf{G^F_{IS5}}$

In this section, we show the equivalence of $\mathbf{G^K_{IS5}}$ and $\mathbf{G^F_{IS5}}$, thus establishing the equivalence of $\mathbf{G_{IS5}}$ and $\mathbf{G^F_{IS5}}$ as follows:

$$\mathbf{G_{IS5}} \overset{(1)}{\Longleftrightarrow} \mathbf{G^I_{IS5}} \overset{(2)}{\Longleftrightarrow} \mathbf{G^K_{IS5}} \overset{(3)}{\Longleftrightarrow} \mathbf{G^F_{IS5}}$$

where (1) is proved by Theorems D.1 and D.5, (2) by Theorems E.1 and E.3, and (3) by Theorems F.1 and F.2.

Theorem F.1 (Soundness of $\mathbf{G^F_{IS5}}$ with respect to $\mathbf{G^K_{IS5}}$).

(1) If $\Sigma\vdash\Delta\vdash\Gamma\longmapsto C$ then $\Sigma\vdash\Delta\vdash\Gamma\longrightarrow_K C$.
(2) If $\Sigma\vdash\Delta\vdash\Gamma\rhd A\longmapsto C$ then $\Sigma\vdash\Delta\vdash\Gamma\rhd A\longrightarrow_K C$.
(3) If $\Sigma\vdash\Delta\vdash\Gamma\rhd\rhd A\longmapsto C$ then $\Sigma\vdash\Delta\vdash\Gamma\rhd\rhd A\longrightarrow_K C$.
(4) If $\Sigma;\Gamma'\rhd A\vdash\Delta\vdash\Gamma\longmapsto C$ then $\Sigma;\Gamma'\rhd A\vdash\Delta\vdash\Gamma\longrightarrow_K C$.
(5) If $\Sigma;\Gamma'\rhd\rhd A\vdash\Delta\vdash\Gamma\longmapsto C$ then $\Sigma;\Gamma'\rhd\rhd A\vdash\Delta\vdash\Gamma\longrightarrow_K C$.

Proof. We give a translation function f that converts a $\mathbf{G^F_{IS5}}$ derivation into a $\mathbf{G^K_{IS5}}$ derivation. Since the inference rules of the two systems are in one-to-one correspondence (except for the rule $\supset\mathsf{R}_b$), the translation is quite straightforward, that is, f simply replaces every $\mathbf{G^F_{IS5}}$ rule application with the corresponding $\mathbf{G^K_{IS5}}$ rule application. For a $\mathbf{G^F_{IS5}}$ $\supset\mathsf{R}_b$ rule application, we replace it with a $\mathbf{G^K_{IS5}}$ $\supset\mathsf{R}$ rule application. As a $\mathbf{G^K_{IS5}}$ sequent may include any redundant formulas in the contexts, the proof tree obtained by f is valid in $\mathbf{G^K_{IS5}}$. □

The statement of the completeness theorem is more complex since $\mathbf{G^K_{IS5}}$ allows redundant rule applications, which add into the contexts formulas that are not strictly necessary. Such rule applications are not allowed in $\mathbf{G^F_{IS5}}$ and thus we simply ignore them when translating a $\mathbf{G^K_{IS5}}$ derivation into a $\mathbf{G^F_{IS5}}$ derivation.

Theorem F.2 (Completeness of $\mathbf{G^F_{IS5}}$ with respect to $\mathbf{G^K_{IS5}}$).

(1) If $\Sigma\vdash\Delta\vdash\Gamma\longrightarrow_K C$ then $\Sigma_f\vdash\Delta_f\vdash\Gamma_f\longmapsto C$.
(2) If $\Sigma\vdash\Delta\vdash\Gamma\rhd A\longrightarrow_K C$ then either $\Sigma_f\vdash\Delta_f\vdash\Gamma_f\longmapsto C$ or $\Sigma_f\vdash\Delta_f\vdash\Gamma_f\rhd A\longmapsto C$.
(3) If $\Sigma\vdash\Delta\vdash\Gamma\rhd\rhd A\longrightarrow_K C$ then either $\Sigma_f\vdash\Delta_f\vdash\Gamma_f\longmapsto C$ or $\Sigma_f\vdash\Delta_f\vdash\Gamma_f\rhd\rhd A\longmapsto C$.
(4) If $\Sigma;\Gamma'\rhd A\vdash\Delta\vdash\Gamma\longrightarrow_K C$ then either $\Sigma_f;\Gamma'_f\vdash\Delta_f\vdash\Gamma_f\longmapsto C$ or $\Sigma_f;\Gamma'_f\rhd A\vdash\Delta_f\vdash\Gamma_f\longmapsto C$.
(5) If $\Sigma;\Gamma'\rhd\rhd A\vdash\Delta\vdash\Gamma\longrightarrow_K C$ then either $\Sigma_f;\Gamma'_f\vdash\Delta_f\vdash\Gamma_f\longmapsto C$ or $\Sigma_f;\Gamma'_f\rhd\rhd A\vdash\Delta_f\vdash\Gamma_f\longmapsto C$.

where $\Sigma_f\subseteq\Sigma$, $\Delta_f\subseteq\Delta$, $\Gamma_f\subseteq\Gamma$, $\Gamma'_f\subseteq\Gamma'$.

Proof. By simultaneous induction on the structure of the given derivation. We give some interesting cases; the other cases are similarly proved.

Case $\dfrac{}{\Sigma \vdash \Delta \vdash \Gamma \rhd p \longrightarrow_K p}\ \mathsf{Id}^{local}$

(1) $\cdot \vdash \cdot \vdash \cdot \rhd p \longmapsto p$ by Id^{local}

Case $\dfrac{}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd p \longrightarrow_K p}\ \mathsf{Id}^{global}$

(1) $\cdot \vdash \cdot \vdash \cdot \rhd \rhd p \longmapsto p$ by Id^{global}

Case $\dfrac{}{\Sigma \vdash \Delta \vdash \Gamma \rhd \rhd \bot \longrightarrow_K C}\ \bot\mathsf{L}_a^{global}$

(1) $\cdot \vdash \cdot \vdash \cdot \rhd \rhd \bot \longmapsto C$ by $\bot\mathsf{L}_a^{global}$

Case $\dfrac{\Sigma; \Gamma', A \rhd A \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma', A \vdash \Delta \vdash \Gamma \longrightarrow_K C}\ \mathsf{ch}^{frame}$

Subcase:

(1) $\Sigma_f; \Gamma'_f \vdash \Delta_f \vdash \Gamma_f \longmapsto C$ by I.H.

Subcase:

(1) $\Sigma_f; \Gamma'_f \rhd A \vdash \Delta_f \vdash \Gamma_f \longmapsto C$ by I.H.

(2) $\Sigma_f; \Gamma'_f, A \vdash \Delta_f \vdash \Gamma_f \longmapsto C$ by ch^{frame}

Case $\dfrac{\Sigma; \Gamma \vdash \Delta \vdash \cdot \longrightarrow_K A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K \Box A}\ \Box\mathsf{R}$

(1) $\Sigma_f; \Gamma_f \vdash \Delta_f \vdash \cdot \longmapsto A$ by I.H.

(2) $\Sigma_f \vdash \Delta_f \vdash \Gamma_f \longmapsto \Box A$ by $\Box\mathsf{R}$

Case $\dfrac{\Sigma \vdash \Delta, A \vdash \Gamma \longrightarrow_K C}{\Sigma \vdash \Delta \vdash \Gamma \rhd \Box A \longrightarrow_K C}\ \Box\mathsf{L}^{local}$

Subcase:

(1) $\Sigma_f \vdash \Delta_f \vdash \Gamma_f \longmapsto C \qquad (A \notin \Delta_f)$ by I.H.

Subcase:

(1) $\Sigma_f \vdash \Delta_f, A \vdash \Gamma_f \longmapsto C$ by I.H.

(2) $\Sigma_f \vdash \Delta_f \vdash \Gamma_f \rhd \Box A \longmapsto C$ by $\Box\mathsf{L}^{local}$

Case $\dfrac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow_K A}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow_K \Diamond A}\ \Diamond\mathsf{R}_b$

(1) $\Sigma_f; \Gamma_f \vdash \Delta_f \vdash \Gamma'_f \longmapsto A$ by I.H.

(2) $\Sigma_f; \Gamma'_f \vdash \Delta_f \vdash \Gamma_f \longmapsto \Diamond A$ by $\Diamond\mathsf{R}_b$

Case $\dfrac{\Sigma; \Gamma'; A \vdash \Delta \vdash \Gamma \longrightarrow_K C}{\Sigma; \Gamma' \rhd \rhd \Diamond A \vdash \Delta \vdash \Gamma \longrightarrow_K C}\ \Diamond\mathsf{L}_b^{global}$

Subcase:

(1) $\Sigma_f; \Gamma'_f \vdash \Delta_f \vdash \Gamma_f \longmapsto C$ by I.H.

Subcase:

(1) $\Sigma_f; \Gamma'_f; A \vdash \Delta_f \vdash \Gamma_f \longmapsto C$ by I.H.

(2) $\Sigma_f; \Gamma'_f \rhd \rhd \Diamond A \vdash \Delta_f \vdash \Gamma_f \longmapsto C$ by $\Diamond \mathsf{L}_b^{global}$

□

G A BOOKKEEPING METHOD FOR $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}$(L) AND ITS EQUIVALENCE TO $\mathbf{G}_{\mathbf{IS5}}$

In this section, we introduce a bookkeeping method similar to the one in [11]. It is mainly developed for the purpose of ensuring the termination of our backward proof search, and can be further optimized for implementation. During the backward proof search, each $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ sequent is coupled with the following nine bookkeeping sets: $\Box\mathsf{L}^{\mathcal{R}}$, $\Diamond\mathsf{L}^{\mathcal{R}}$, $\vee\mathsf{L}^{\mathcal{R}}$, $\bot\mathsf{L}^{\mathcal{R}}$, $\mathsf{Id}^{\mathcal{R}}$, $\Box\mathsf{L}^{\mathcal{T}}$, $\Diamond\mathsf{L}^{\mathcal{T}}$, $\Box\mathsf{R}^{\mathcal{R}}$, $\Diamond\mathsf{R}^{\mathcal{R}}$. Each bookkeeping set records the labels of the corresponding rule instances that have been applied so far and thus should not be applied again. $\Box\mathsf{L}^{\mathcal{R}}$, $\Diamond\mathsf{L}^{\mathcal{R}}$, $\vee\mathsf{L}^{\mathcal{R}}$, $\bot\mathsf{L}^{\mathcal{R}}$ and $\mathsf{Id}^{\mathcal{R}}$ are used for the five types of derived rule instances, and $\Box\mathsf{R}^{\mathcal{R}}$ and $\Diamond\mathsf{R}^{\mathcal{R}}$ for $\Box$R and $\Diamond$R rule instances. The labels contained in $\Box\mathsf{L}^{\mathcal{T}}$ and $\Diamond\mathsf{L}^{\mathcal{T}}$ indicate on which rule instance's twigs the current sequent is located. Such rule instances become available again if the contexts of the sequents along the derivation rooted at the twigs are modified. In addition, we maintain a list path of sequents from the current one to the root (*i.e.*, the initial sequent $\cdot \vdash \cdot \vdash \cdot \longrightarrow L$) to detect a cycle.

Below we provide an operational description of how each bookkeeping set is updated in the premise sequents of each $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ inference rule (instance). Here we consider only those derived rule instances analyzing a formula in the local context; the bookkeeping sets update is similar for the cases analyzing an accessible context in the frame.

(1) For a $\Box$L derived rule instance ($A \notin \Delta$)

$$\frac{\overbrace{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_1 \quad \cdots \quad \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_n}^{twigs} \quad \overbrace{\Sigma \vdash \Delta, A \vdash \Gamma_j \longrightarrow C}^{trunk}}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C}\ \Box\mathsf{L}(i,j)$$

with a principal formula i to be applicable, the $\Box\mathsf{L}^{\mathcal{R}}$ set of the conclusion must not contain (i,j). For an application of this rule instance, we update the bookkeeping sets of the premises as follows. We first copy the bookkeeping sets of the conclusion into each premise. Then, in each trunk sequent:

(a) $\Box\mathsf{L}^{\mathcal{R}} \leftarrow (\Box\mathsf{L}^{\mathcal{R}} \setminus \Box\mathsf{L}^{\mathcal{T}}) \cup \{(i,j)\}$
(b) $\Diamond\mathsf{L}^{\mathcal{R}} \leftarrow (\Diamond\mathsf{L}^{\mathcal{R}} \setminus \Diamond\mathsf{L}^{\mathcal{T}})$
(c) Empty the remaining bookkeeping sets.

In each twig sequent:

(a) $\Box\mathsf{L}^{\mathcal{R}} \leftarrow \Box\mathsf{L}^{\mathcal{R}} \cup \{(i,j)\}$
(b) $\Box\mathsf{L}^{\mathcal{T}} \leftarrow \Box\mathsf{L}^{\mathcal{T}} \cup \{(i,j)\}$
(c) The remaining bookkeeping sets are unchanged.

(2) For a $\Diamond$L derived rule instance ($\nexists\Gamma, A \in \Gamma \in \Sigma$)

$$\frac{\overbrace{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_1 \quad \cdots \quad \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_n}^{twigs} \quad \overbrace{\Sigma; \{A\}_i \vdash \Delta \vdash \Gamma_j \longrightarrow C}^{trunk}}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C}\ \Diamond\mathsf{L}(i,j)$$

with a principal formula i to be applicable, the $\Diamond\mathsf{L}^{\mathcal{R}}$ set of the conclusion must not contain (i,j). For an application of this rule instance, we first copy the bookkeeping sets of the conclusion into each premise. Then, in each trunk sequent:

(a) $\Box\mathsf{L}^{\mathcal{R}} \leftarrow \Box\mathsf{L}^{\mathcal{R}} \setminus \Box\mathsf{L}^{\mathcal{T}}$
(b) $\Diamond\mathsf{L}^{\mathcal{R}} \leftarrow (\Diamond\mathsf{L}^{\mathcal{R}} \setminus \Diamond\mathsf{L}^{\mathcal{T}}) \cup \{(i,j)\}$

(c) Empty the remaining bookkeeping sets.

In the twig sequents:

(a) $\Diamond\mathsf{L}^{\mathcal{R}} \longleftarrow \Diamond\mathsf{L}^{\mathcal{R}} \cup \{(i,j)\}$

(b) $\Diamond\mathsf{L}^{\mathcal{T}} \longleftarrow \Diamond\mathsf{L}^{\mathcal{T}} \cup \{(i,j)\}$

(c) The remaining bookkeeping sets are unchanged.

(3) For a $\vee\mathsf{L}$ derived rule instance $(A_1, A_2 \notin \Gamma)$

$$\frac{\overbrace{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_1 \quad \cdots \quad \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_n}^{twigs} \quad \overbrace{\Sigma \vdash \Delta \vdash \Gamma_j, A_1 \longrightarrow C \quad \Sigma \vdash \Delta \vdash \Gamma_j, A_2 \longrightarrow C}^{trunks}}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C} \;\vee\mathsf{L}(i,j)$$

with a principal formula i to be applicable, the $\vee\mathsf{L}^{\mathcal{R}}$ set of the conclusion must not contain (i,j). For an application of this rule instance, we first copy the bookkeeping sets of the conclusion into each premise. Then, in each trunk sequent:

(a) $\Box\mathsf{L}^{\mathcal{R}} \longleftarrow \Box\mathsf{L}^{\mathcal{R}} \setminus \Box\mathsf{L}^{\mathcal{T}}$

(b) $\Diamond\mathsf{L}^{\mathcal{R}} \longleftarrow \Diamond\mathsf{L}^{\mathcal{R}} \setminus \Diamond\mathsf{L}^{\mathcal{T}}$

(c) $\vee\mathsf{L}^{\mathcal{R}} \longleftarrow \{(i,j)\}$

(d) Empty the remaining bookkeeping sets.

In each twig sequent:

(a) $\vee\mathsf{L}^{\mathcal{R}} \longleftarrow \vee\mathsf{L}^{\mathcal{R}} \cup \{(i,j)\}$

(b) The remaining bookkeeping sets are unchanged.

(4) For a $\bot\mathsf{L}$ derived rule instance

$$\frac{\overbrace{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_1 \quad \cdots \quad \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_n}^{twigs}}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C} \;\bot\mathsf{L}(i,j)$$

with a principal formula i to be applicable, the $\bot\mathsf{L}^{\mathcal{R}}$ set of the conclusion must not contain (i,j). For an application of this rule instance, we first copy the bookkeeping sets of the conclusion into each premise. Then, in each premise:

(a) $\bot\mathsf{L}^{\mathcal{R}} \longleftarrow \bot\mathsf{L}^{\mathcal{R}} \cup \{(i,j)\}$

(b) The remaining bookkeeping sets are unchanged.

(5) For an Id derived rule instance

$$\frac{\overbrace{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_1 \quad \cdots \quad \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_n}^{twigs}}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C} \;\mathsf{Id}(i,j)$$

with a principal formula i to be applicable, the $\mathsf{Id}^{\mathcal{R}}$ set of the conclusion must not contain (i,j). For an application of this rule instance, we first copy the bookkeeping sets of the conclusion into each premise. Then, in each premise:

(a) $\mathsf{Id}^{\mathcal{R}} \longleftarrow \mathsf{Id}^{\mathcal{R}} \cup \{(i,j)\}$

(b) The remaining bookkeeping sets are unchanged.

(6) For a $\Box\mathsf{R}$ rule instance

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \{\cdot\}_j \longrightarrow A}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow \Box A} \;\Box\mathsf{R}$$

where j is the label of $\Box A$ to be applicable, (i) the $\Box\mathsf{R}^{\mathcal{R}}$ set of the conclusion must not contain $\Box A$ and (ii) $\Sigma' \vdash \Delta \vdash \Gamma \longrightarrow \Box A \notin$ path where $\Sigma = \Sigma'; \Gamma$. For an application of this inference rule, we first copy the bookkeeping sets of the conclusion into the premise. Then:

(a) $\Box\mathsf{L}^{\mathcal{R}} \longleftarrow \Box\mathsf{L}^{\mathcal{R}} \setminus \Box\mathsf{L}^{\mathcal{T}}$

(b) $\Diamond\mathsf{L}^{\mathcal{R}} \leftarrow \Diamond\mathsf{L}^{\mathcal{R}} \setminus \Diamond\mathsf{L}^{\mathcal{T}}$
(c) $\Box\mathsf{R}^{\mathcal{R}} \leftarrow \Box\mathsf{R}^{\mathcal{R}} \cup \{\Box A\}$
(d) $\Diamond\mathsf{R}^{\mathcal{R}}$ is unchanged.
(e) Empty the remaining bookkeeping sets.

(7) For the rule $\Diamond\mathsf{R}_b$ to be applicable, the $\Diamond\mathsf{R}^{\mathcal{R}}$ set of the conclusion must not contain $\Diamond A$.

$$\frac{\Sigma; \Gamma \vdash \Delta \vdash \Gamma' \longrightarrow A}{\Sigma; \Gamma' \vdash \Delta \vdash \Gamma \longrightarrow \Diamond A} \ \Diamond\mathsf{R}_b$$

For an application of this inference rule, we first copy the bookkeeping sets of the conclusion into the premise. Then:
(a) $\Box\mathsf{L}^{\mathcal{R}} \leftarrow \Box\mathsf{L}^{\mathcal{R}} \setminus \Box\mathsf{L}^{\mathcal{T}}$
(b) $\Diamond\mathsf{L}^{\mathcal{R}} \leftarrow \Diamond\mathsf{L}^{\mathcal{R}} \setminus \Diamond\mathsf{L}^{\mathcal{T}}$
(c) $\Box\mathsf{R}^{\mathcal{R}}$ is unchanged.
(d) $\Diamond\mathsf{R}^{\mathcal{R}} \leftarrow \Diamond\mathsf{R}^{\mathcal{R}} \cup \{\Diamond A\}$
(e) Empty the remaining bookkeeping sets.

(8) When applying the rule $\supset\mathsf{R}$, we consider two cases.
(8-1) $A \notin \Gamma$:

$$\frac{\Sigma \vdash \Delta \vdash \Gamma, A \longrightarrow B}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \supset B} \ \supset\mathsf{R}_a$$

For an application of this inference rule, we first copy the bookkeeping sets of the conclusion into the premise. Then:
(a) $\Box\mathsf{L}^{\mathcal{R}} \leftarrow \Box\mathsf{L}^{\mathcal{R}} \setminus \Box\mathsf{L}^{\mathcal{T}}$
(b) $\Diamond\mathsf{L}^{\mathcal{R}} \leftarrow \Diamond\mathsf{L}^{\mathcal{R}} \setminus \Diamond\mathsf{L}^{\mathcal{T}}$
(c) Empty the remaining bookkeeping sets.

(8-2) $A \in \Gamma$:

$$\frac{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow B}{\Sigma \vdash \Delta \vdash \Gamma \longrightarrow A \supset B} \ \supset\mathsf{R}_b$$

For an application of this inference rule, the bookkeeping sets of the premise are the same as those of the conclusion.

(9) As for an application of the remaining inference rules ($\wedge\mathsf{R}$, $\vee\mathsf{R}_l$, $\vee\mathsf{R}_r$, or $\Diamond\mathsf{R}_a$), the bookkeeping sets of the premises are the same as those of the conclusion.

Theorem G.1 (Termination of Backward Proof Search in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$). *Backward proof search in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ is guaranteed to terminate.*

Proof. We define the size of a $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ sequent

$$\Sigma \vdash \Delta \vdash \Gamma \longrightarrow C \parallel \Box\mathsf{L}^{\mathcal{R}}; \Diamond\mathsf{L}^{\mathcal{R}}; \vee\mathsf{L}^{\mathcal{R}}; \bot\mathsf{L}^{\mathcal{R}}; \mathsf{Id}^{\mathcal{R}}; \Box\mathsf{L}^{\mathcal{T}}; \Diamond\mathsf{L}^{\mathcal{T}}; \Box\mathsf{R}^{\mathcal{R}}; \Diamond\mathsf{R}^{\mathcal{R}}$$

as 11-tuple $\langle s_1, \ldots, s_{11} \rangle$ where $s_1, \ldots, s_{11}$ are defined as follows.

$$\begin{array}{rclrcl} s_1 &=& |(\Box\mathsf{L}^{\mathcal{R}} \setminus \Box\mathsf{L}^{\mathcal{T}})^c| & s_2 &=& |(\Diamond\mathsf{L}^{\mathcal{R}} \setminus \Diamond\mathsf{L}^{\mathcal{T}})^c| \\ s_3 &=& \mathcal{F} - |\Sigma| - |\Gamma| & s_4 &=& |(\Box\mathsf{R}^{\mathcal{R}})^c| \\ s_5 &=& |(\Diamond\mathsf{R}^{\mathcal{R}})^c| & s_6 &=& |(\Box\mathsf{L}^{\mathcal{R}})^c| \\ s_7 &=& |(\Diamond\mathsf{L}^{\mathcal{R}})^c| & s_8 &=& |(\vee\mathsf{L}^{\mathcal{R}})^c| \\ s_9 &=& |(\bot\mathsf{L}^{\mathcal{R}})^c| & s_{10} &=& |(\mathsf{Id}^{\mathcal{R}})^c| \\ s_{11} &=& \mathit{size}(C) & & & \end{array}$$

where S^c is the complement of the set S and $\mathit{size}(C)$ denotes the size of a formula C, which is defined inductively in the usual way. $|\Sigma| = |\Gamma_1| + \cdots + |\Gamma_n|$ when $\Sigma = \Gamma_1; \ldots; \Gamma_n$ and $|\Gamma|$ denotes the number of distinct formulas in Γ. $\mathcal{F}$ is the number of distinct negative subformulas of the query formula L

times the maximum number of accessible contexts that can be created during the backward proof search for L. The number of accessible contexts is bounded by the number of applications of $\Diamond$L and $\Box$R which can be applied only a finite number of times because of the bookkeeping sets and the loop detection method using path. Each element of the size tuple of a sequent is greater than or equal to zero and $\langle s_1, \ldots, s_{11} \rangle < \langle t_1, \ldots, t_{11} \rangle$ if and only if there exists i such that $s_i < t_i$ and $s_j = t_j$ for $j < i$. The size of every premise of every $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ rule application is smaller than the size of the conclusion, and thus our backward proof search terminates. □

Now we show the equivalence of $\mathbf{G}_{\mathbf{IS5}}$ and $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ by showing that $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ is equivalent to $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ which is again equivalent to $\mathbf{G}_{\mathbf{IS5}}$.

Theorem G.2 (Soundness of $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ with respect to $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$).
If $\mathcal{D} :: \Sigma \vdash \Delta \vdash \Gamma \longrightarrow C$ in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ then $\mathcal{E} :: \Sigma \vdash \Delta \vdash \Gamma \longrightarrow_K C$ in $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$.

Proof. By induction on the structure of the given derivation. The derivation $\mathcal{E}$ can be easily obtained by simply ignoring the bookkeeping sets used in the derivation $\mathcal{D}$ and converting each $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ rule application into a corresponding $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ rule application. Note that the derived rules used in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ are also valid in $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ because of the way they are constructed. Moreover, the right rules of $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ are the same as those of $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ except for the $\supset \mathsf{R}_b$ rule. For each $\supset \mathsf{R}_b$ rule application used in $\mathcal{D}$, we simply convert it into a $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ $\supset$R rule application. □

Theorem G.3 (Completeness of $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ with respect to $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$).
If $\mathcal{D} :: \cdot \vdash \cdot \vdash \cdot \longrightarrow_K L$ in $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ then $\cdot \vdash \cdot \vdash \cdot \longrightarrow L$ in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$, with the initial bookkeeping sets generated from the query formula L.

Proof. We will show how to translate a $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ derivation into a $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ derivation. First, note that every $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ right rule is also available in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$. Moreover, every $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ left rule occurs in a focused thread, which can be replaced by the corresponding $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ derived rule. Then, the main difficulty in the translation is to eliminate a rule application that violates a side condition of the bookkeeping sets. Below we show how to eliminate a violating rule instance; by repeatedly applying this elimination procedure, we build a valid $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ derivation from a $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ derivation.

Case 1: An application of a $\Box$L derived rule instance

$$\cfrac{\overset{\mathcal{D}_1}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_1} \quad \cdots \quad \overset{\mathcal{D}_n}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_n} \quad \overset{\mathcal{E}}{\Sigma \vdash \Delta, A \vdash \Gamma_j \longrightarrow C}}{\underset{\mathcal{F}}{\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C}} \ \Box\mathsf{L}(i,j)$$

violates a side condition of the bookkeeping sets. Then, either

(1) $A \in \Delta$ or
(2) the $\Box\mathsf{L}^{\mathcal{R}}$ set of the conclusion contains (i, j).

In the first case, by applying the contraction property of $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ to $\mathcal{E}$, we can obtain a proof of $\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C$. In the second case, (i, j) must have been added into the branch's $\Box\mathsf{L}^{\mathcal{R}}$ set by an application of $\Box\mathsf{L}(i, j)$ in $\mathcal{F}$. Consider the closest such application.

Subcase 1-1: The current branch is on one of the twig premises of the closest application of $\Box\mathsf{L}(i,j)$.

$$\mathcal{F} = \dfrac{\cdots\quad \begin{array}{c}\Sigma\vdash\Delta\vdash\Gamma_j\longrightarrow C\\ \mathcal{F}_k \\ \Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow B_k\end{array}\quad\cdots\quad \begin{array}{c}\mathcal{E}'\\ \Sigma'\vdash\Delta',A\vdash\Gamma'_j\longrightarrow C'\end{array}}{\Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow C'}\ \Box\mathsf{L}(i,j)$$
$$\mathcal{H}$$

The application of $\Box\mathsf{L}(i,j)$ adds (i,j) to the $\Box\mathsf{L}^{\mathcal{R}}$ and $\Box\mathsf{L}^{\mathcal{T}}$ sets of each twig. Every rule that changes any context, except for $\Box\mathsf{L}(i,j)$, removes (i,j) from the $\Box\mathsf{L}^{\mathcal{R}}$ set of its premise whose context is changed. Therefore, every sequent in the path from $\Sigma\vdash\Delta\vdash\Gamma_j\longrightarrow C$ to $\Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow B_k$ in the partial derivation $\mathcal{F}_k$ should have the same set of contexts, *i.e.*, $\Sigma=\Sigma'$, $\Delta=\Delta'$, $\Gamma_j=\Gamma'_j$; otherwise, the upper application of $\Box\mathsf{L}(i,j)$ would be valid. Therefore, we can replace the violating segment by

$$\dfrac{\cdots\quad \begin{array}{c}\mathcal{D}_k \\ \Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow B_k\end{array}\quad\cdots\quad \begin{array}{c}\mathcal{E}'\\ \Sigma'\vdash\Delta',A\vdash\Gamma'_j\longrightarrow C'\end{array}}{\Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow C'}\ \Box\mathsf{L}(i,j)$$
$$\mathcal{H}$$

Subcase 1-2: The current branch is on the trunk premise of the closest application of $\Box\mathsf{L}(i,j)$.

$$\mathcal{F} = \dfrac{\begin{array}{c}\mathcal{F}_1 \\ \Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow B_1\end{array}\quad\cdots\quad \begin{array}{c}\mathcal{F}_n \\ \Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow B_n\end{array}\quad \begin{array}{c}\Sigma\vdash\Delta\vdash\Gamma_j\longrightarrow C\\ \mathcal{G}\\ \Sigma'\vdash\Delta',A\vdash\Gamma'_j\longrightarrow C'\end{array}}{\Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow C'}\ \Box\mathsf{L}(i,j)$$
$$\mathcal{H}$$

No rule removes a formula from the global context, so $A\in\Delta$. Then, we can replace the violating segment by

$$\dfrac{\begin{array}{c}\mathcal{F}_1 \\ \Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow B_1\end{array}\quad\cdots\quad \begin{array}{c}\mathcal{F}_n \\ \Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow B_n\end{array}\quad \begin{array}{c}\mathcal{E}'\\ \Sigma\vdash\Delta\vdash\Gamma_j\longrightarrow C\\ \mathcal{G}\\ \Sigma'\vdash\Delta',A\vdash\Gamma'_j\longrightarrow C'\end{array}}{\Sigma'\vdash\Delta'\vdash\Gamma'_j\longrightarrow C'}\ \Box\mathsf{L}(i,j)$$
$$\mathcal{H}$$

where $\mathcal{E}'$ is obtained by applying the contraction property of $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ to $\mathcal{E}$.

Case 2: An application of a $\Diamond\mathsf{L}$ derived rule instance

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Sigma\vdash\Delta\vdash\Gamma_j\longrightarrow B_1\end{array}\quad\cdots\quad \begin{array}{c}\mathcal{D}_n \\ \Sigma\vdash\Delta\vdash\Gamma_j\longrightarrow B_n\end{array}\quad \begin{array}{c}\mathcal{E}\\ \Sigma;\{A\}_i\vdash\Delta\vdash\Gamma_j\longrightarrow C\end{array}}{\Sigma\vdash\Delta\vdash\Gamma_j\longrightarrow C}\ \Diamond\mathsf{L}(i,j)$$
$$\mathcal{F}$$

violates a side condition of the bookkeeping sets. Then, either

(1) $\exists\Gamma, A\in\Gamma\in\Sigma$ or
(2) the $\Diamond\mathsf{L}^{\mathcal{R}}$ set of the conclusion contains (i,j).

In the first case, by applying the contraction property of $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ to $\mathcal{E}$, we can obtain a proof of $\Sigma\vdash\Delta\vdash\Gamma_j\longrightarrow C$. In the second case, (i,j) must have been added into the branch's $\Diamond\mathsf{L}^{\mathcal{R}}$ set by an application of $\Diamond\mathsf{L}(i,j)$ in $\mathcal{F}$. Consider the closest such application.

Subcase 2-1: The current branch is on one of the twig premises of the closest application of $\Diamond\mathsf{L}(i,j)$.

$$\mathcal{F} = \dfrac{\cdots \quad \begin{array}{c}\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C\\ \mathcal{F}_k \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{E}'\\ \Sigma'; \{A\}_i \vdash \Delta' \vdash \Gamma'_j \longrightarrow C'\end{array}}{\begin{array}{c}\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow C' \\ \mathcal{H}\end{array}}\ \Diamond\mathsf{L}(i,j)$$

The application of $\Diamond\mathsf{L}(i,j)$ adds (i,j) to the $\Diamond\mathsf{L}^{\mathcal{R}}$ and $\Diamond\mathsf{L}^{\mathcal{T}}$ sets of each twig. Every rule that changes any context, except for $\Diamond\mathsf{L}(i,j)$, removes (i,j) from the $\Diamond\mathsf{L}^{\mathcal{R}}$ set of its premise whose context is changed. Therefore, every sequent in the path from $\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C$ to $\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k$ in the partial derivation $\mathcal{F}_k$ should have the same set of contexts, *i.e.*, $\Sigma = \Sigma'$, $\Delta' = \Delta$, $\Gamma_j = \Gamma'_j$; otherwise, the upper application of $\Diamond\mathsf{L}(i,j)$ would be valid. Therefore, we can replace the violating segment by

$$\dfrac{\cdots \quad \begin{array}{c}\mathcal{D}_k \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{E}'\\ \Sigma'; \{A\}_i \vdash \Delta' \vdash \Gamma'_j \longrightarrow C'\end{array}}{\begin{array}{c}\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow C' \\ \mathcal{H}\end{array}}\ \Diamond\mathsf{L}(i,j)$$

Subcase 2-2: The current branch is on the trunk premise of the closest application of $\Diamond\mathsf{L}(i,j)$.

$$\mathcal{F} = \dfrac{\begin{array}{c}\mathcal{F}_1 \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{F}_n \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_n\end{array} \quad \begin{array}{c}\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C\\ \mathcal{G}\\ \Sigma'; \{A\}_i \vdash \Delta' \vdash \Gamma'_j \longrightarrow C'\end{array}}{\begin{array}{c}\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow C' \\ \mathcal{H}\end{array}}\ \Diamond\mathsf{L}(i,j)$$

No rule removes an accessible context from the frame, so $\exists\Gamma_i, \{A\}_i \subseteq \Gamma_i \in \Sigma;\Gamma_j$. Then, we can replace the violating segment by

$$\dfrac{\begin{array}{c}\mathcal{F}_1 \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{F}_n \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_n\end{array} \quad \begin{array}{c}\mathcal{E}'\\ \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C\\ \mathcal{G}\\ \Sigma'; \{A\}_i \vdash \Delta' \vdash \Gamma'_j \longrightarrow C'\end{array}}{\begin{array}{c}\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow C' \\ \mathcal{H}\end{array}}\ \Diamond\mathsf{L}(i,j)$$

where $\mathcal{E}'$ is obtained by applying the contraction property of $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ to $\mathcal{E}$.

Case 3: An application of a $\vee\mathsf{L}$ derived rule instance

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n \\ \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_n\end{array} \quad \begin{array}{c}\mathcal{E}_1 \\ \Sigma \vdash \Delta \vdash \Gamma_j, A_1 \longrightarrow C\end{array} \quad \begin{array}{c}\mathcal{E}_2 \\ \Sigma \vdash \Delta \vdash \Gamma_j, A_2 \longrightarrow C\end{array}}{\begin{array}{c}\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C \\ \mathcal{F}\end{array}}\ \vee\mathsf{L}(i,j)$$

violates a side condition of the corresponding rule in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$. Then, either

(1) $A_1 \in \Gamma_j$,
(2) $A_2 \in \Gamma_j$, or
(3) the $\vee\mathsf{L}^{\mathcal{R}}$ set of the conclusion contains (i,j).

In the first two cases, by applying the contraction property of $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ to either $\mathcal{E}_1$ or $\mathcal{E}_2$, we can obtain a proof of $\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C$, which eliminates the violating application of $\vee\mathsf{L}(i,j)$. In the last case, (i,j) must have been added into the branch's $\vee\mathsf{L}^{\mathcal{R}}$ set by an application of $\vee\mathsf{L}(i,j)$ in $\mathcal{F}$. Consider the closest such application.

Subcase 3-1: The current branch is on one of the twig premises of the closest application of $\vee\mathsf{L}(i,j)$.

$$\mathcal{F} = \dfrac{\cdots \quad \begin{array}{c}\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C \\ \mathcal{F}_k \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{G}_1 \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j, A_1 \longrightarrow C'\end{array} \quad \begin{array}{c}\mathcal{G}_2 \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j, A_2 \longrightarrow C'\end{array}}{\begin{array}{c}\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow C' \\ \mathcal{H}\end{array}} \vee\mathsf{L}(i,j)$$

The application of $\vee\mathsf{L}(i,j)$ adds (i,j) to the $\vee\mathsf{L}^{\mathcal{R}}$ set of each twig. Every rule that changes any context, except for $\vee\mathsf{L}(i,j)$, removes (i,j) from the $\vee\mathsf{L}^{\mathcal{R}}$ set of its premise whose context is changed. Therefore, every sequent in the path from $\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C$ to $\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k$ in the partial derivation $\mathcal{F}_k$ should have the same set of contexts, *i.e.*, $\Sigma = \Sigma'$, $\Delta' = \Delta$, $\Gamma_j = \Gamma'_j$; otherwise, the upper application of $\vee\mathsf{L}(i,j)$ would be valid. Therefore, we can replace the violating segment by

$$\dfrac{\cdots \quad \begin{array}{c}\mathcal{D}_k \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{G}_1 \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j, A_1 \longrightarrow C'\end{array} \quad \begin{array}{c}\mathcal{G}_2 \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j, A_2 \longrightarrow C'\end{array}}{\begin{array}{c}\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow C' \\ \mathcal{H}\end{array}} \vee\mathsf{L}(i,j)$$

Subcase 3-2: The current branch is on one of the trunk premises of the closest application of $\vee\mathsf{L}(i,j)$. Suppose it is on the sequent $\Sigma' \vdash \Delta' \vdash \Gamma'_j, A_1 \longrightarrow C'$; the other case is similar.

$$\mathcal{F} = \dfrac{\cdots \quad \begin{array}{c}\mathcal{F}_k \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k\end{array} \cdots \quad \begin{array}{c}\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C \\ \mathcal{G} \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j, A_1 \longrightarrow C'\end{array} \quad \begin{array}{c}\mathcal{G}_2 \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j, A_2 \longrightarrow C'\end{array}}{\begin{array}{c}\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow C' \\ \mathcal{H}\end{array}} \vee\mathsf{L}(i,j)$$

No rule removes a formula from the local context and the rules $\Box\mathsf{R}$ and $\Diamond\mathsf{R}_b$ have not been applied in $\mathcal{G}$, so $A_1 \in \Gamma_j$. Then, we can replace the violating segment by

$$\dfrac{\cdots \quad \begin{array}{c}\mathcal{F}_k \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k\end{array} \cdots \quad \begin{array}{c}\mathcal{E}'_1 \\ \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C \\ \mathcal{G} \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j, A_1 \longrightarrow C'\end{array} \quad \begin{array}{c}\mathcal{G}_2 \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j, A_2 \longrightarrow C'\end{array}}{\begin{array}{c}\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow C' \\ \mathcal{H}\end{array}} \vee\mathsf{L}(i,j)$$

where $\mathcal{E}'$ is obtained by applying the contraction property of $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ to $\mathcal{E}_1$.

Case 4: An application of a $\bot\mathsf{L}$ derived rule instance

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_1\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n \\ \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_n\end{array}}{\begin{array}{c}\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C \\ \mathcal{F}\end{array}} \bot\mathsf{L}(i,j)$$

violates a side condition of the bookkeeping sets; in other words, the $\bot\mathsf{L}^{\mathcal{R}}$ set of the conclusion contains (i,j). Then, (i,j) must have been added into the branch's $\bot\mathsf{L}^{\mathcal{R}}$ set by an application of

$\bot\mathsf{L}(i,j)$ in $\mathcal{F}$. Consider the closest such application.

$$\mathcal{F} = \dfrac{\begin{matrix} & \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C & \\ \mathcal{F}_1 & \mathcal{F}_k & \mathcal{F}_n \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_1 \quad \cdots & \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k & \cdots \quad \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_n \end{matrix}}{\begin{matrix}\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow C' \\ \mathcal{H}\end{matrix}} \ \bot\mathsf{L}(i,j)$$

The application of $\bot\mathsf{L}(i,j)$ adds (i,j) to the $\bot\mathsf{L}^{\mathcal{R}}$ set of each twig. Every rule that changes any context empties the $\bot\mathsf{L}^{\mathcal{R}}$ set of its premise whose context is changed. Therefore, every sequent in the path from $\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C$ to $\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k$ in the partial derivation $\mathcal{F}_k$ should have the same set of contexts, *i.e.*, $\Sigma = \Sigma'$, $\Delta' = \Delta$, $\Gamma_j = \Gamma'_j$; otherwise, the upper application of $\bot\mathsf{L}(i,j)$ would be valid. Therefore, we can replace the violating segment by

$$\dfrac{\begin{matrix} \mathcal{F}_1 & \mathcal{D}_k & \mathcal{F}_n \\ \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_1 \quad \cdots & \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_k & \cdots \quad \Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow B_n \end{matrix}}{\begin{matrix}\Sigma' \vdash \Delta' \vdash \Gamma'_j \longrightarrow C' \\ \mathcal{H}\end{matrix}} \ \bot\mathsf{L}(i,j)$$

Case 5: An application of an Id derived rule instance

$$\dfrac{\begin{matrix} \mathcal{D}_1 & & \mathcal{D}_n \\ \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_1 & \cdots & \Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow B_n \end{matrix}}{\begin{matrix}\Sigma \vdash \Delta \vdash \Gamma_j \longrightarrow C \\ \mathcal{F}\end{matrix}} \ \mathsf{Id}(i,j)$$

violates a side condition of the bookkeeping sets; in other words, the $\mathsf{Id}^{\mathcal{R}}$ set of the conclusion contains (i,j). The translation is similar to Case 4.

Case 6: An application of a $\Box\mathsf{R}$ rule instance

$$\dfrac{\begin{matrix} \mathcal{D} \\ \Sigma;\Gamma \vdash \Delta \vdash \{\cdot\}_j \longrightarrow A \end{matrix}}{\begin{matrix}\Sigma \vdash \Delta \vdash \Gamma \longrightarrow \Box A \\ \mathcal{F}\end{matrix}} \ \Box\mathsf{R}$$

where j is the label of $\Box A$ violates a side condition of the bookkeeping sets. Then, either

(1) the $\Box\mathsf{R}^{\mathcal{R}}$ set of the conclusion contains $\Box A$, or
(2) $\Sigma' \vdash \Delta \vdash \Gamma \longrightarrow \Box A \in$ path where $\Sigma = \Sigma';\Gamma$.

In the first case, $\Box A$ must have been added into the branch's $\Box\mathsf{R}^{\mathcal{R}}$ set by an application of $\Box\mathsf{R}$ in $\mathcal{F}$. Consider the closest such application.

$$\mathcal{F} = \dfrac{\begin{matrix} \Sigma \vdash \Delta \vdash \Gamma \longrightarrow \Box A \\ \mathcal{G} \\ \Sigma';\Gamma' \vdash \Delta' \vdash \{\cdot\}_j \longrightarrow A \end{matrix}}{\begin{matrix}\Sigma' \vdash \Delta' \vdash \Gamma' \longrightarrow \Box A \\ \mathcal{H}\end{matrix}} \ \Box\mathsf{R}$$

Every rule that changes any context, except for $\Box\mathsf{R}$ and $\Diamond\mathsf{R}_b$, empties the $\Box\mathsf{R}^{\mathcal{R}}$ set of its premise whose context is changed, so $\Sigma;\Gamma = \Sigma';\Gamma'$ modulo reordering and $\Delta = \Delta'$. Therefore, we can replace the violating segment by

$$\dfrac{\overset{\mathcal{D}}{\Sigma'; \Gamma' \vdash \Delta' \vdash \{\cdot\}_j \longrightarrow A}}{\underset{\mathcal{H}}{\Sigma' \vdash \Delta' \vdash \Gamma' \longrightarrow \Box A}} \ \Box\mathsf{R}$$

In the second case, the derivation looks like the following:

$$\dfrac{\overset{\mathcal{D}}{\Sigma; \Gamma \vdash \Delta \vdash \{\cdot\}_j \longrightarrow A}}{\mathcal{F} = \begin{cases} \Sigma \vdash \Delta \vdash \Gamma \longrightarrow \Box A \\ \quad \mathcal{G} \\ \Sigma' \vdash \Delta \vdash \Gamma \longrightarrow \Box A \\ \quad \mathcal{H} \end{cases}} \ \Box\mathsf{R}$$

We can replace the violating segment by

$$\dfrac{\overset{\mathcal{D}'}{\Sigma'; \Gamma \vdash \Delta \vdash \{\cdot\}_j \longrightarrow A}}{\underset{\mathcal{H}}{\Sigma' \vdash \Delta \vdash \Gamma \longrightarrow \Box A}} \ \Box\mathsf{R}$$

where $\mathcal{D}'$ is obtained by applying the contraction property of $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ to $\mathcal{D}$.

Case 7: An application of a $\Diamond\mathsf{R}_b$ rule instance

$$\dfrac{\overset{\mathcal{D}}{\Sigma; \Gamma_1 \vdash \Delta \vdash \Gamma_2 \longrightarrow A}}{\underset{\mathcal{F}}{\Sigma; \Gamma_2 \vdash \Delta \vdash \Gamma_1 \longrightarrow \Diamond A}} \ \Diamond\mathsf{R}_b$$

violates a side condition of the bookkeeping sets; in other words, the $\Diamond\mathsf{R}^{\mathcal{R}}$ set of the conclusion contains $\Diamond A$. Then, $\Diamond A$ must have been added into the branch's $\Diamond\mathsf{R}^{\mathcal{R}}$ set by an application of $\Diamond\mathsf{R}_b$ in $\mathcal{F}$. Consider the closest such application.

$$\mathcal{F} = \begin{array}{c} \Sigma; \Gamma_2 \vdash \Delta \vdash \Gamma_1 \longrightarrow \Diamond A \\ \mathcal{G} \\ \dfrac{\Sigma'; \Gamma_i \vdash \Delta' \vdash \Gamma_j \longrightarrow A}{\Sigma'; \Gamma_j \vdash \Delta' \vdash \Gamma_i \longrightarrow \Diamond A} \ \Diamond\mathsf{R}_b \\ \mathcal{H} \end{array}$$

Every rule that changes any context, except for $\Box\mathsf{R}$ and $\Diamond\mathsf{R}_b$, empties the $\Diamond\mathsf{R}^{\mathcal{R}}$ set of its premise whose context is changed, so $\Sigma; \Gamma_1; \Gamma_2 = \Sigma'; \Gamma_i; \Gamma_j$ modulo reordering and $\Delta = \Delta'$. Therefore, we can replace the violating segment by

$$\dfrac{\overset{\mathcal{D}}{\Sigma; \Gamma_1 \vdash \Delta \vdash \Gamma_2 \longrightarrow A}}{\underset{\mathcal{H}}{\Sigma'; \Gamma_j \vdash \Delta' \vdash \Gamma_i \longrightarrow \Diamond A}} \ \Diamond\mathsf{R}_b$$

Remaining Cases: Applications of the remaining $\mathbf{G}^{\mathbf{K}}_{\mathbf{IS5}}$ right rules need not be translated as they are also valid in $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$. Note that the $\mathbf{G}^{\mathbf{B}}_{\mathbf{IS5}}(L)$ right rules $\supset\mathsf{R}_a$, $\supset\mathsf{R}_b$, $\wedge\mathsf{R}$, $\vee\mathsf{R}_l$, $\vee\mathsf{R}_r$ and $\Diamond\mathsf{R}_a$ have no side conditions.

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