CSE-490 Assignment 3 - *Normalization*
(100 points)

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Due at 1:00am, October 16 (Monday)

In this assignment, you will prove the normalization property of the natural deduction system with the implication connective $\supset$ only. To help the instructor grade this assignment, you are required to typeset your answers using LaTeX.

- For this assignment, you may work with up to one other student.
- For this assignment, you are free to consult outside material (other than Course Notes).
- On the cover page of your assignment, you must state one of the following:
  - I worked alone on this assignment.
  - I worked with *(your classmate with whom you discussed this assignment)* on this assignment.
- On the cover page of your assignment, you must state one of the following:
  - I used only the course material provided on the course webpage.
  - I used the following outside material to complete this assignment: *(sources of outside material)*.

1 Preliminaries

Lexicographical ordering

A lexicographical ordering $\succ$ on pairs (m, n) of natural numbers is defined as follows, where $m \geq 0$ and $n \geq 0$ are assumed:

$$(m, n) \succ (m', n') \text{ iff. either (1) } m > m', \text{ or (2) } m = m' \text{ and } n > n'.$$

$\succ$ is an example of a *well-founded ordering* in that there is no infinite descending chain

$$(m_0, n_0) \succ (m_1, n_1) \succ \cdots \succ (m_i, n_i) \succ \cdots.$$

Size of a proposition

We define the size $\mathsf{size}(A)$ of a proposition A as follows, where P denotes an atomic proposition:

$$\begin{array}{rcl} \mathsf{size}(P) & = & 1 \\ \mathsf{size}(A \supset B) & = & \max(\mathsf{size}(A), \mathsf{size}(B)) + 1 \end{array}$$

Degree of a detour

Consider a detour in which an introduction rule is used to deduce A *true* to which a corresponding elimination rule is applied. The degree of the detour is defined as $\mathsf{size}(A)$.

Degree of a proof

We define the degree of a proof $\mathcal{D}$ as the maximum of the degrees of all the detours in $\mathcal{D}$. If $\mathcal{D}$ contains no detours, we let its degree be zero.

Size of a proof

We define the size $\mathsf{size}(\mathcal{D})$ of a proof $\mathcal{D}$ as a pair (d, n) where d is the degree of $\mathcal{D}$ and n is the number of detours of degree d in $\mathcal{D}$. If $\mathcal{D}$ contains no detours, we let $\mathsf{size}(\mathcal{D})$ be $(0, 0)$. Note that if $\mathsf{size}(\mathcal{D}) = (d, n)$ and $d > 0$, then $n > 0$.

2 Normalization

We consider the following fragment of propositional logic:

$$\dfrac{\begin{array}{c}\overline{A\ true}^{\,x}\\ \vdots\\ B\ true\end{array}}{A \supset B\ true}\ \supset\mathsf{I}^x \qquad\qquad \dfrac{A \supset B\ true \quad A\ true}{B\ true}\ \supset\mathsf{E}$$

Implicit in the rule $\supset\mathsf{I}^x$ is that the hypothetical proof in the premise may contain multiple occurrences of the hypothesis $\overline{A\ true}^{\,x}$, since we may use the same hypothesis zero times, just once, or as many times as necessary.

Lemma 2.1. *For a proof $\mathcal{D}$ of A true with some detours (*i.e.*, a proof $\mathcal{D}$ of A true that is not normal), there exists another proof $\mathcal{D}'$ of A true such that $\mathsf{size}(\mathcal{D}) \succ \mathsf{size}(\mathcal{D}')$.*

Proof. Fill in the proof. You should give a formal proof using induction. (80 points) □

You want to check that your proof of Lemma 2.1 actually gives an *algorithm* for transforming a proof with detours into another proof so that its size decreases.

Theorem 2.2. *For every proof of A true, there is a finite sequence of local reductions that lead to a normal proof of A true.*

Proof. Fill in the proof. Use the property of well-founded orderings. (20 points) □

Submission instruction

Download `hw3.tex`, `defns.tex`, `proof.sty` from the course webpage and copy them to your working directory.

Fill in `hw3.tex` with your answers.

Produce a PostScript file `hw3.ps`. Copy both `hw3.ps` and `hw3.tex` to the following directory (if your Hemos id is `foo`):

```
/afs/postech.ac.kr/class/cse/cs490/handin/hw3/foo/
```

Note that we do not accept PDF files! A sample session might look like:

```
[gla:23 ] latex hw3.tex
[gla:24 ] dvips -o hw3.ps hw3.dvi
[gla:25 ] cp hw3.ps /afs/postech.ac.kr/class/cse/cs490/handin/hw3/<Your Hemos ID>/
[gla:26 ] cp hw3.tex /afs/postech.ac.kr/class/cse/cs490/handin/hw3/<Your Hemos ID>/
```