Chapter 1

First-Order Logic

1.1 Terms

$$\text{term} \quad t \quad ::= \quad x \mid y \mid \cdots \mid a \mid b \mid \cdots \mid f(t_1, \cdots, t_n) \mid c$$

- x is called a *term variable* which ranges over the set of terms.
- a is called a *parameter* and denotes an arbitrary/unspecified term about which we can make no assumption.
- f is called a *function symbol*. $f(t_1, \cdots, t_n)$ is a term where f is a function symbol of arity n and $t_1, \cdots, t_n$ are its arguments.
- f is not a function in that $f(t_1, \cdots, t_n)$ does not reduce to another term; $f(t_1, \cdots, t_n)$ is a term in itself.
- A *constant* c is a function symbol with zero arity. We abbreviate $c()$ as c.
- Terms are not to be confused with proof terms: a term can be any kind of object (*e.g.*, natural number, tree, boolean value, student name, *etc*) whereas a proof term is a particular kind of object representing a proof in logic.

Example

$$\text{natural number} \quad t \quad ::= \quad \mathbf{0} \mid \mathbf{s}(t)$$

1.2 Propositions in first-order logic

$$\text{proposition} \quad A \quad ::= \quad P(t_1, \cdots, t_n) \mid \cdots \mid \forall x.A \mid \exists x.A$$

$$\frac{}{P(t_1, \cdots, t_n)\ prop}\ P\mathsf{F} \qquad \frac{A\ prop}{\forall x.A\ prop}\ \forall\mathsf{F} \qquad \frac{A\ prop}{\exists x.A\ prop}\ \exists\mathsf{F}$$

- P is called a *predicate symbol*. A *predicate* $P(t_1, \cdots, t_n)$ is a proposition which expresses a certain relation between terms $t_1, \cdots, t_n$. Thus we may think of predicates as propositions about objects.
- A propositional constant P is a predicate symbol with zero arity. We abbreviate $P()$ as P.
- $\forall x.A$ uses a *universal quantifier* $\forall$ to introduce a term variable x. Roughly speaking, the truth of $\forall x.A$ means that A is true for "every" term x.

- $\exists x.A$ uses a *existential quantifier* $\exists$ to introduce a term variable x. Roughly speaking, the truth of $\exists x.A$ means that A is true for "some" term x.
- Quantifiers $\forall$ and $\exists$ have the lowest operator precedence. For example, $\forall x.A \supset B$ is understood as $\forall x.(A \supset B)$; similarly $\exists x.A \supset B$ is understood as $\exists x.(A \supset B)$.
- As quantifiers introduce term variables, there arises a need for substitutions for term variables in propositions or proofs. We write $[t/x]A$ for the result of substituting t for x in proposition A. Similarly we write $[t/x]\mathcal{D}$ for the result of substituting t for x throughout proof $\mathcal{D}$.
- we write $[t/a]A$ and $[t/a]\mathcal{D}$ for the result of substituting t for *parameter* a in A and $\mathcal{D}$, respectively.
- Variable captures never occur in first-order logic: in a substitution $[t/x]A$ or $[t/x]\mathcal{D}$, term t is always closed, *i.e.*, it does not contain free term variables.

Example

$$\text{proposition} \quad A \quad ::= \quad Nat(t) \mid Eq(t,t) \mid \cdots$$

1.3 Universal quantifier

$$\frac{[a/x]A\ true}{\forall x.A\ true}\ \forall\mathsf{I}^a \qquad \frac{\forall x.A\ true}{[t/x]A\ true}\ \forall\mathsf{E}$$

- We may think of $\forall x.A$ as an infinite conjunction

  $$[t_1/x]A \wedge [t_2/x]A \wedge \cdots \wedge [t_i/x]A \wedge \cdots$$

  where $t_1, t_2, \cdots, t_i, \cdots$ enumerate all terms available.
- Parameter a denotes an an arbitrary term about which we can make no assumption, and the proof of $[a/x]A\ true$ in the premise of the rule $\forall\mathsf{I}^a$ is parameteric in a.
- In the rule $\forall\mathsf{I}^a$, parameter a must be fresh in that it is not found in any undischarged hypothesis in the proof of the premise.
- We may think of parameter a in the rule $\forall\mathsf{I}^a$ as an "arbitrary" term that is specific to the rule $\forall\mathsf{I}^a$ or that becomes fixed when the premise $[a/x]A\ true$ is decided. Hence using parameter a again for another instance of the rule $\forall\mathsf{I}$ results in a wrong proof. In the proof shown below, parameter a is specific to the rule $\forall\mathsf{I}^a$ in the bottom, as indicated by the partial proof in the left. From the point of view of the rule $\forall\mathsf{I}^a$ in the top, therefore, the same parameter a cannot be read as an "arbitrary" term again because it can be read an "arbitrary" term only with respect to the rule $\forall\mathsf{I}^a$ in the bottom. Note that in the rule $\forall\mathsf{I}^a$ in the top, parameter a is already found in an undischarged hypothesis $\overline{Nat(a)\ true}^{\,w}$.

$$\frac{\begin{array}{c}\vdots\\ \forall y.Nat(a) \supset Nat(y)\ true\end{array}}{\forall x.\forall y.Nat(x) \supset Nat(y)\ true}\ \forall\mathsf{I}^a \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Nat(a)\ true}^{\,w}}{\forall x.Nat(x)\ true}\ \forall\mathsf{I}^a\ \textit{(wrong)}}{Nat(b)\ true}\ \forall\mathsf{E}}{Nat(a) \supset Nat(b)\ true}\ \supset\mathsf{I}^w}{\forall y.Nat(a) \supset Nat(y)\ true}\ \forall\mathsf{I}^b}{\forall x.\forall y.Nat(x) \supset Nat(y)\ true}\ \forall\mathsf{I}^a$$

- In the rule $\forall\mathsf{E}$, we may use any term for t — term variable, parameter, function symbol, constant, *etc.*

- Here is an example of a proof involving universal quantifiers which uses $[a/x](A \wedge B) = [a/x]A \wedge [a/x]B$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x. A \wedge B}^{\,w}}{[a/x](A \wedge B)\ true}\ \forall E}{[a/x]A\ true}\ \wedge E_L}{\forall x. A\ true}\ \forall I^a \qquad \dfrac{\dfrac{\dfrac{\overline{\forall x. A \wedge B}^{\,w}}{[a/x](A \wedge B)\ true}\ \forall E}{[a/x]B\ true}\ \wedge E_R}{\forall x. B\ true}\ \forall I^a}{\dfrac{(\forall x. A) \wedge (\forall x. B)\ true}{(\forall x. A \wedge B) \supset (\forall x. A) \wedge (\forall x. B)\ true}\ \supset I^w}\ \wedge I
$$

Hypothetical judgments

$$
\frac{\Gamma \vdash [a/x]A\ true}{\Gamma \vdash \forall x. A\ true}\ \forall I^a \qquad \frac{\Gamma \vdash \forall x. A\ true}{\Gamma \vdash [t/x]A\ true}\ \forall E
$$

Local reduction and local expansion

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ [a/x]A\ true\end{array}}{\forall x. A\ true}\ \forall I^a}{[t/x]A\ true}\ \forall E \quad \Longrightarrow_R \quad \begin{array}{c}[t/a]\mathcal{D}\\ [t/x]A\ true\end{array}
$$

$$
\begin{array}{c}\mathcal{E}\\ \forall x. A\ true\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{E}\\ \forall x. A\ true\end{array}}{[a/x]A\ true}\ \forall E}{\forall x. A\ true}\ \forall I^a
$$

Normal and neutral judgments

$$
\frac{[a/x]A\uparrow}{\forall x. A\uparrow}\ \forall I^a \qquad \frac{\forall x. A\downarrow}{[t/x]A\downarrow}\ \forall E
$$

$$
\frac{\Gamma_\downarrow \vdash [a/x]A\uparrow}{\Gamma_\downarrow \vdash \forall x. A\uparrow}\ \forall I^a \qquad \frac{\Gamma \vdash \forall x. A\downarrow}{\Gamma \vdash [t/x]A\downarrow}\ \forall E
$$

1.4 Existential quantifier

$$
\frac{[t/x]A\ true}{\exists x. A\ true}\ \exists I \qquad \dfrac{\exists x. A\ true \qquad \begin{array}{c}\overline{[a/x]A\ true}^{\,w}\\ \vdots\\ C\ true\end{array}}{C\ true}\ \exists E^{a,w}
$$

- We may think of $\forall x. A$ as an infinite disjunction

$$
[t_1/x]A \vee [t_2/x]A \vee \cdots \vee [t_i/x]A \vee \cdots
$$

  where $t_1, t_2, \cdots, t_i, \cdots$ enumerate all terms available.

- The rule $\exists$I says that in order to prove $\exists x. A\ true$, we must present a concrete term, or a *witness*, t such that $[t/x]A\ true$ is provable. It is not enough to state that there exists such a term without actually knowing what it is.

- The necessity of a witness in the rule $\exists$I is a distinguishing feature of constructive logic. In contrast, a proof of $\exists x.A\ true$ in classical logic only needs to show that there exists a term t, *which may or may not be known*, such that $[t/x]A\ true$ is provable. In other words, a proof of $\exists x.A\ true$ essentially shows that it cannot happen that there exists no term t such that $[t/x]A\ true$ is provable. As a consequence, $\exists x.A$ is no different from $\neg\forall x.\neg A$ in classical logic.

- In the rule $\exists$E, we do not know the witness for the proof of $\exists x.A\ true$ and thus cannot make any assumption about it. Therefore, if we are to make use of a proof of $\exists x.A\ true$, we have to introduce a fresh parameter a.

- In the rule $\exists$E, parameter a must not be found in A or any undischarged hypothesis. In particular, it must not be found in C; otherwise the rule ends up with a conclusion that makes too strong an assumption about the witness, namely that it can be an arbitrary term! For example, the following proof draws a nonsensical conclusion that an arbitrary term is equal to a natural number $\mathbf{0}$, as it allows parameter a to appear in the conclusion:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Nat(\mathbf{0})\ true} \quad \dfrac{\dfrac{}{\forall x.Eq(x,x)\ true}}{Eq(\mathbf{0},\mathbf{0})\ true}\forall\mathsf{E}}{Nat(\mathbf{0}) \wedge Eq(\mathbf{0},\mathbf{0})\ true}\wedge\mathsf{I}}{\exists x.Nat(x)\wedge Eq(x,\mathbf{0})\ true}\exists\mathsf{I} \qquad \dfrac{\dfrac{}{Nat(a)\wedge Eq(a,\mathbf{0})\ true}w}{Eq(a,\mathbf{0})\ true}\wedge\mathsf{E_R}}{Eq(a,\mathbf{0})\ true}\exists\mathsf{E}^{a,w}$$

Example

- $\exists x.\neg A \supset \neg\forall x.A\ true$ is provable. Intuitively a proof of $\exists x.\neg A\ true$ gives us a witness t such that $[t/x]\neg A\ true$ is provable, and we can use t to refute $\forall x.A\ true$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x.\neg A\ true}w \qquad \dfrac{\dfrac{}{[a/x]\neg A\ true}y \quad \dfrac{\dfrac{}{\forall x.A\ true}z}{[a/x]A\ true}\forall\mathsf{E}}{\bot\ true}\neg\mathsf{E}}{\bot\ true}\exists\mathsf{E}^{a,y}}{\neg\forall x.A\ true}\neg\mathsf{I}^z}{\exists x.\neg A \supset \neg\forall x.A\ true}\supset\mathsf{I}^w$$

- $\neg\forall x.A \supset \exists x.\neg A\ true$ is not provable. Intuitively a proof of $\exists x.\neg A\ true$ requires a witness t such that $[t/x]\neg A\ true$ is provable, but no proof of $\neg\forall x.A\ true$ gives such a witness.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\forall x.A\ true}w \quad \overset{?}{\forall x.A\ true}}{\bot\ true}\neg\mathsf{E}}{\exists x.\neg A\ true}\bot\mathsf{E}}{\neg\forall x.A \supset \exists x.\neg A\ true}\supset\mathsf{I}^w$$

- $(\forall x.A) \supset (\exists x.A)\ true$ is *not* provable. The reason is that although $\forall x.A\ true$ states that $[t/x]A\ true$ is provable for any term t, it does not decide a concrete term t such that $[t/x]A\ true$ is provable. In particular, if the set of terms is empty, $\forall x.A\ true$ holds trivially (because there is no term), but $\exists x.A\ true$ never holds because it is impossible to choose a term t for x, regardless of proposition A.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.A\ true}w}{[t/x]A\ true?}\forall\mathsf{E}}{\exists x.A\ true}\exists\mathsf{I}}{(\forall x.A) \supset (\exists x.A)\ true}\supset\mathsf{I}^w$$

- On the other hand, $\forall y.(\forall x.A) \supset (\exists x.A)\ \mathit{true}$ *is* provable even if y does not occur free in A. The difference from the previous example is that $\forall y$ allows us to make an assumption that the set of terms is not empty. In the proof shown below, paramter a denotes an arbitrary term in the set of terms, and its presence implies that the set of terms in not empty.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.A\ \mathit{true}}\,w}{[a/x]A\ \mathit{true}}\,\forall\mathsf{E}}{\exists x.A\ \mathit{true}}\,\exists\mathsf{I}}{(\forall x.A) \supset (\exists x.A)\ \mathit{true}}\,\supset\mathsf{I}^w}{\forall y.(\forall x.A) \supset (\exists x.A)\ \mathit{true}}\,\forall\mathsf{I}^a$$

- The two examples above illustrate that in constructive logic, $\forall x.A$ is not equivalent to A even if x does not occur free in A at all: $\forall x.A$ asserts A on the assumption that the set of terms is not empty, whereas A without a universal quantifier cannot exploit such an assumption.

Hypothetical judgments

$$\dfrac{\Gamma \vdash [t/x]A\ \mathit{true}}{\Gamma \vdash \exists x.A\ \mathit{true}}\,\exists\mathsf{I} \qquad \dfrac{\Gamma \vdash \exists x.A\ \mathit{true} \quad \Gamma, [a/x]A\ \mathit{true} \vdash C\ \mathit{true}}{\Gamma \vdash C\ \mathit{true}}\,\exists\mathsf{E}^a$$

Local reduction and local expansion

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ [t/x]A\ \mathit{true}\end{array}}{\exists x.A\ \mathit{true}}\,\exists\mathsf{I} \qquad \left.\begin{array}{c}\overline{[a/x]A\ \mathit{true}}^{\,w}\\ \vdots\\ C\ \mathit{true}\end{array}\right\}\mathcal{E}}{C\ \mathit{true}}\,\exists\mathsf{E}^{a,w} \quad \Longrightarrow_R \quad \left.\begin{array}{c}\mathcal{D}\\ [t/x]A\ \mathit{true}\\ \vdots\\ C\ \mathit{true}\end{array}\right\}[t/a]\mathcal{E}$$

$$\begin{array}{c}\mathcal{E}\\ \exists x.A\ \mathit{true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{E}\\ \exists x.A\ \mathit{true}\end{array} \quad \dfrac{\overline{[a/x]A\ \mathit{true}}^{\,w}}{\exists x.A\ \mathit{true}}\,\exists\mathsf{I}}{\exists x.A\ \mathit{true}}\,\exists\mathsf{E}^{a,w}$$

In the local reduction, $[t/a]\mathcal{E}$ does not affect the conclusion $C\ \mathit{true}$ because parameter a does not appear in C. $[t/x]A\ \mathit{true}$ is obtained from $[t/a][a/x]A\ \mathit{true}$.

Normal and neutral judgments

$$\dfrac{[t/x]A\uparrow}{\exists x.A\uparrow}\,\exists\mathsf{I} \qquad \dfrac{\exists x.A\downarrow \qquad \begin{array}{c}\overline{[a/x]A\downarrow}^{\,w}\\ \vdots\\ C\uparrow\end{array}}{C\uparrow}\,\exists\mathsf{E}^{a,w}$$

$$\dfrac{\Gamma_\downarrow \vdash [t/x]A\uparrow}{\Gamma_\downarrow \vdash \exists x.A\uparrow}\,\exists\mathsf{I} \qquad \dfrac{\Gamma_\downarrow \vdash \exists x.A\downarrow \quad \Gamma_\downarrow, [a/x]A\downarrow \vdash C\uparrow}{\Gamma_\downarrow \vdash C\uparrow}\,\exists\mathsf{E}^a$$

1.5 Examples

Axioms

$$\frac{}{Nat(\mathbf{0})\ true}\ Zero \qquad \frac{}{\forall x.Nat(x) \supset Nat(\mathbf{s}(x))\ true}\ Succ$$

$$\frac{}{\forall x.Eq(x,x)\ true}\ Eq_i \qquad \frac{}{\forall x.\forall y.\forall z.(Eq(x,y) \wedge Eq(x,z)) \supset Eq(y,z)\ true}\ Eq_t$$

$$\frac{}{\forall x.Lt(x,\mathbf{s}(x))\ true}\ Lt_s \qquad \frac{}{\forall x.\forall y.Eq(x,y) \supset \neg Lt(x,y)\ true}\ Lt_\neg$$

Theorems

Proof of $\forall x.Nat(x) \supset (\exists y.Nat(y) \wedge Eq(x,y))\ true$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Nat(a)\ true}\,z \quad \dfrac{\dfrac{}{\forall x.Eq(x,x)\ true}\,Eq_i}{Eq(a,a)\ true}\,\forall\mathsf{E}}{Nat(a) \wedge Eq(a,a)\ true}\,\wedge\mathsf{I}}{\exists y.Nat(y) \wedge Eq(a,y)\ true}\,\exists\mathsf{I}}{Nat(a) \supset (\exists y.Nat(y) \wedge Eq(a,y))\ true}\,\supset\mathsf{I}^z}{\forall x.Nat(x) \supset (\exists y.Nat(y) \wedge Eq(x,y))\ true}\,\forall\mathsf{I}^a$$

Proof of $\forall x.\forall y.Eq(x,y) \supset Eq(y,x)\ true$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\forall y.\forall z.(Eq(x,y) \wedge Eq(x,z)) \supset Eq(y,z)\ true}\,Eq_t}{\forall y.\forall z.(Eq(a,y) \wedge Eq(a,z)) \supset Eq(y,z)\ true}\,\forall\mathsf{E}}{\forall z.(Eq(a,b) \wedge Eq(a,z)) \supset Eq(b,z)\ true}\,\forall\mathsf{E}}{(Eq(a,b) \wedge Eq(a,a)) \supset Eq(b,a)\ true}\,\forall\mathsf{E} \quad \dfrac{\dfrac{}{Eq(a,b)\ true}\,w \quad \dfrac{\dfrac{}{\forall x.Eq(x,x)\ true}\,Eq_i}{Eq(a,a)\ true}\,\forall\mathsf{E}}{Eq(a,b) \wedge Eq(a,a)\ true}\,\wedge\mathsf{I}}{Eq(b,a)\ true}\,\supset\mathsf{E}}{Eq(a,b) \supset Eq(b,a)\ true}\,\supset\mathsf{I}^w}{\forall y.Eq(a,y) \supset Eq(y,a)\ true}\,\forall\mathsf{I}^b}{\forall x.\forall y.Eq(x,y) \supset Eq(y,x)\ true}\,\forall\mathsf{I}^a$$

Proof of $\neg\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathbf{s}(\mathbf{0}))\ true$:

$$\mathcal{D} \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\forall y.\forall z.(Eq(x,y) \wedge Eq(x,z)) \supset Eq(y,z)\ true}\,Eq_t}{\forall y.\forall z.(Eq(a,y) \wedge Eq(a,z)) \supset Eq(y,z)\ true}\,\forall\mathsf{E}}{\forall z.(Eq(a,\mathbf{0}) \wedge Eq(a,z)) \supset Eq(\mathbf{0},z)\ true}\,\forall\mathsf{E}}{(Eq(a,\mathbf{0}) \wedge Eq(a,\mathbf{s}(\mathbf{0}))) \supset Eq(\mathbf{0},\mathbf{s}(\mathbf{0}))\ true}\,\forall\mathsf{E} \quad \dfrac{}{Eq(a,\mathbf{0}) \wedge Eq(a,\mathbf{s}(\mathbf{0}))\ true}\,z}{Eq(\mathbf{0},\mathbf{s}(\mathbf{0}))\ true}\,\supset\mathsf{E}$$

$$\dfrac{\dfrac{\dfrac{}{\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathbf{s}(\mathbf{0}))\ true}\,w \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\forall y.Eq(x,y) \supset \neg Lt(x,y)\ true}\,Lt_\neg}{\forall y.Eq(\mathbf{0},y) \supset \neg Lt(\mathbf{0},y)\ true}\,\forall\mathsf{E}}{Eq(\mathbf{0},\mathbf{s}(\mathbf{0})) \supset \neg Lt(\mathbf{0},\mathbf{s}(\mathbf{0}))\ true}\,\forall\mathsf{E} \quad \begin{matrix}\mathcal{D}\\ Eq(\mathbf{0},\mathbf{s}(\mathbf{0}))\ true\end{matrix}}{\neg Lt(\mathbf{0},\mathbf{s}(\mathbf{0}))\ true}\,\supset\mathsf{E} \quad \dfrac{\dfrac{}{\forall x.Lt(x,\mathbf{s}(x))\ true}\,Lt_s}{Lt(\mathbf{0},\mathbf{s}(\mathbf{0}))\ true}\,\forall\mathsf{E}}{\bot\ true}\,\neg\mathsf{E}}{\bot\ true}\,\exists\mathsf{E}^{a,z}}{\neg\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathbf{s}(\mathbf{0}))\ true}\,\neg\mathsf{I}^w$$

1.6 Proof terms

$$\text{proof term} \quad M \quad ::= \quad \cdots \mid \lambda x.\, M \mid M\ t \mid \langle t, M\rangle \mid \mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ M$$

A substitution $[t/x]M$ is defined as usual. Note that $[t/x]M$ may need a substitution $[t/x]A$ if x is found within a type A in M.

$$\frac{[a/x]M : [a/x]A}{\lambda x.\, M : \forall x.A}\ \forall\mathsf{I}^a \qquad \frac{M : \forall x.A}{M\ t : [t/x]A}\ \forall\mathsf{E}$$

$$\frac{M : [t/x]A}{\langle t, M\rangle : \exists x.A}\ \exists\mathsf{I} \qquad \frac{M : \exists x.A \qquad \begin{array}{c}\overline{w : [a/x]A}\\ \vdots \\ [a/x]N : C\end{array}}{\mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ N : C}\ \exists\mathsf{E}^a$$

Hypothetical judgments

$$\frac{\Gamma \vdash [a/x]M : [a/x]A}{\Gamma \vdash \lambda x.\, M : \forall x.A}\ \forall\mathsf{I}^a \qquad \frac{\Gamma \vdash M : \forall x.A}{\Gamma \vdash M\ t : [t/x]A}\ \forall\mathsf{E}$$

$$\frac{\Gamma \vdash M : [t/x]A}{\Gamma \vdash \langle t, M\rangle : \exists x.A}\ \exists\mathsf{I} \qquad \frac{\Gamma \vdash M : \exists x.A \quad \Gamma, w : [a/x]A \vdash [a/x]N : C}{\Gamma \vdash \mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ N : C}\ \exists\mathsf{E}^a$$

Local reduction and expansion

Univeral quantifier:

$$\cfrac{\cfrac{[a/x]M : [a/x]A}{\lambda x.\, M : \forall x.A}\ \forall\mathsf{I}^a}{(\lambda x.\, M)\ t : [t/x]A}\ \forall\mathsf{E} \qquad \Longrightarrow_R \qquad [t/a][a/x]M : [t/a][a/x]A$$

$$M : \forall x.A \qquad \Longrightarrow_E \qquad \cfrac{\cfrac{M : \forall x.A}{M\ a : [a/x]A}\ \forall\mathsf{E}}{\lambda x.\, M\ x : \forall x.A}\ \forall\mathsf{I}^a\ \textit{(where } M\ a = [a/x](M\ x)\textit{)}$$

$$\begin{array}{lll} (\lambda x.\, M)\ t & \Longrightarrow_R & [t/x]M \\ M : \forall x.A & \Longrightarrow_E & \lambda x.\, M\ x \qquad \textit{(}x\textit{ is not free in }M\textit{)} \end{array}$$

Existential quantifier:

$$\cfrac{\cfrac{M : [t/x]A}{\langle t, M\rangle : \exists x.A}\ \exists\mathsf{I} \qquad \begin{array}{c}\overline{w : [a/x]A}\\ \vdots \\ [a/x]N : C\end{array}}{\mathsf{let}\ \langle x, w\rangle = \langle t, M\rangle\ \mathsf{in}\ N : C}\ \exists\mathsf{E}^{a,w} \qquad \Longrightarrow_R \qquad \begin{array}{c}[M/w][t/a]w : [t/a][a/x]A\\ \vdots \\ [M/w][t/a][a/x]N : C\end{array}$$

$$M : \exists x.A \qquad \Longrightarrow_E \qquad \cfrac{M : \exists x.A \qquad \cfrac{\overline{w : [a/x]A}}{\langle a, w\rangle : \exists x.A}\ \exists\mathsf{I}}{\mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ \langle x, w\rangle : \exists x.A}\ \exists\mathsf{E}^a\ \textit{(where } \langle a, w\rangle = [a/x]\langle x, w\rangle\textit{)}$$

$$\begin{array}{rll} \mathsf{let}\ \langle x, w\rangle = \langle t, M\rangle\ \mathsf{in}\ N & \Longrightarrow_R & [M/w][t/x]N \\ M : \exists x.A & \Longrightarrow_E & \mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ \langle x, w\rangle \end{array}$$

Terms in normal form

$$\begin{array}{llcl}
\text{elim term} & E & ::= & \cdots \mid E\; t \\
\text{intro term} & I & ::= & \cdots \mid \lambda x.\, I \mid \langle t, I\rangle \mid \mathsf{let}\ \langle x, w\rangle = E\ \mathsf{in}\ I
\end{array}$$

1.6.1 Examples

Axioms

$$\dfrac{}{\mathbf{Nat_0} : Nat(\mathbf{0})}\ Zero \qquad \dfrac{}{\mathbf{Nat_s} : \forall x.Nat(x) \supset Nat(\mathbf{s}(x))}\ Succ$$

$$\dfrac{}{\mathbf{Eq_i} : \forall x.Eq(x,x)}\ Eq_i \qquad \dfrac{}{\mathbf{Eq_t} : \forall x.\forall y.\forall z.(Eq(x,y) \wedge Eq(x,z)) \supset Eq(y,z)}\ Eq_t$$

$$\dfrac{}{\mathbf{Lt_s} : \forall x.Lt(x,\mathbf{s}(x))}\ Lt_s \qquad \dfrac{}{\mathbf{Lt_\neg} : \forall x.\forall y.Eq(x,y) \supset \neg Lt(x,y)}\ Lt_\neg$$

Theorems

Proof term for $\forall x.Nat(x) \supset (\exists y.Nat(y) \wedge Eq(x,y))\ true$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{z : Nat(a)} \qquad \dfrac{\dfrac{}{\mathbf{Eq_i} : \forall x.Eq(x,x)}\ Eq_i}{\mathbf{Eq_i}\ a : Eq(a,a)}\ \forall\mathsf{E}}{(z, \mathbf{Eq_i}\ a) : Nat(a) \wedge Eq(a,a)}\ \wedge\mathsf{I}}{\langle a, (z, \mathbf{Eq_i}\ a)\rangle : \exists y.Nat(y) \wedge Eq(a,y)}\ \exists\mathsf{I}}{\lambda z{:}Nat(a).\, \langle a, (z, \mathbf{Eq_i}\ a)\rangle : Nat(a) \supset (\exists y.Nat(y) \wedge Eq(a,y))}\ \supset\mathsf{I}^z}{\lambda x.\, \lambda z{:}Nat(x).\, \langle x, (z, \mathbf{Eq_i}\ x)\rangle : \forall x.Nat(x) \supset (\exists y.Nat(y) \wedge Eq(x,y))}\ \forall\mathsf{I}^a$$

Proof term for $\forall x.\forall y.Eq(x,y) \supset Eq(y,x)\ true$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{Eq_t} : \forall x.\forall y.\forall z.(Eq(x,y) \wedge Eq(x,z)) \supset Eq(y,z)}\ Eq_t}{\mathbf{Eq_t}\ a : \forall y.\forall z.(Eq(a,y) \wedge Eq(a,z)) \supset Eq(y,z)}\ \forall\mathsf{E}}{\mathbf{Eq_t}\ a\ b : \forall z.(Eq(a,b) \wedge Eq(a,z)) \supset Eq(b,z)}\ \forall\mathsf{E}}{\mathbf{Eq_t}\ a\ b\ a : (Eq(a,b) \wedge Eq(a,a)) \supset Eq(b,a)}\ \forall\mathsf{E} \qquad \dfrac{\dfrac{}{w : Eq(a,b)} \qquad \dfrac{\dfrac{}{\mathbf{Eq_i} : \forall x.Eq(x,x)}\ Eq_i}{\mathbf{Eq_i}\ a : Eq(a,a)}\ \forall\mathsf{E}}{(w, \mathbf{Eq_i}\ a) : Eq(a,b) \wedge Eq(a,a)}\ \wedge\mathsf{I}}{\mathbf{Eq_t}\ a\ b\ a\ (w, \mathbf{Eq_i}\ a) : Eq(b,a)}\ \supset\mathsf{E}}{\lambda w{:}Eq(a,b).\, \mathbf{Eq_t}\ a\ b\ a\ (w, \mathbf{Eq_i}\ a) : Eq(a,b) \supset Eq(b,a)}\ \supset\mathsf{I}^w}{\lambda y.\, \lambda w{:}Eq(a,y).\, \mathbf{Eq_t}\ a\ y\ a\ (w, \mathbf{Eq_i}\ a) : \forall y.Eq(a,y) \supset Eq(y,a)}\ \forall\mathsf{I}^b}{\lambda x.\, \lambda y.\, \lambda w{:}Eq(x,y).\, \mathbf{Eq_t}\ x\ y\ x\ (w, \mathbf{Eq_i}\ x) : \forall x.\forall y.Eq(x,y) \supset Eq(y,x)}\ \forall\mathsf{I}^a$$

Proof term for $\neg\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathbf{s}(\mathbf{0}))\ true$:

$$\mathcal{D} \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{Eq_t} : \forall x.\forall y.\forall z.(Eq(x,y) \wedge Eq(x,z)) \supset Eq(y,z)}\ Eq_t}{\mathbf{Eq_t}\ a : \forall y.\forall z.(Eq(a,y) \wedge Eq(a,z)) \supset Eq(y,z)}\ \forall\mathsf{E}}{\mathbf{Eq_t}\ a\ \mathbf{0} : \forall z.(Eq(a,\mathbf{0}) \wedge Eq(a,z)) \supset Eq(\mathbf{0},z)}\ \forall\mathsf{E}}{\mathbf{Eq_t}\ a\ \mathbf{0}\ \mathbf{s}(\mathbf{0}) : (Eq(a,\mathbf{0}) \wedge Eq(a,\mathbf{s}(\mathbf{0}))) \supset Eq(\mathbf{0},\mathbf{s}(\mathbf{0}))}\ \forall\mathsf{E} \qquad \dfrac{}{z : Eq(a,\mathbf{0}) \wedge Eq(a,\mathbf{s}(\mathbf{0}))}}{\mathbf{Eq_t}\ a\ \mathbf{0}\ \mathbf{s}(\mathbf{0})\ z : Eq(\mathbf{0},\mathbf{s}(\mathbf{0}))}\ \supset\mathsf{E}$$

$$\mathcal{E} \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{}{Lt_\neg : \forall x.\forall y.Eq(x,y) \supset \neg Lt(x,y)}\ Lt_\neg}{Lt_\neg\ \mathbf{0} : \forall y.Eq(\mathbf{0},y) \supset \neg Lt(\mathbf{0},y)}\ \forall\mathsf{E}}{Lt_\neg\ \mathbf{0}\ \mathbf{s}(\mathbf{0}) : Eq(\mathbf{0},\mathbf{s}(\mathbf{0})) \supset \neg Lt(\mathbf{0},\mathbf{s}(\mathbf{0}))}\ \forall\mathsf{E} \qquad \begin{array}{c}\mathcal{D}\\ \mathbf{Eq_t}\ a\ \mathbf{0}\ \mathbf{s}(\mathbf{0})\ z : Eq(\mathbf{0},\mathbf{s}(\mathbf{0}))\end{array}}{(Lt_\neg\ \mathbf{0}\ \mathbf{s}(\mathbf{0}))\ (\mathbf{Eq_t}\ a\ \mathbf{0}\ \mathbf{s}(\mathbf{0})\ z) : \neg Lt(\mathbf{0},\mathbf{s}(\mathbf{0}))}\ \supset\mathsf{E}$$

$$
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\dfrac{
\overline{w : \exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathbf{s}(\mathbf{0}))}
\qquad
\dfrac{
\begin{array}{c}\mathcal{E}\\ (Lt_\neg\ \mathbf{0}\ \mathbf{s}(\mathbf{0}))\ (\mathbf{Eq_t}\ a\ \mathbf{0}\ \mathbf{s}(\mathbf{0})\ z) : \neg Lt(\mathbf{0},\mathbf{s}(\mathbf{0}))\end{array}
\quad
\dfrac{\dfrac{}{\mathbf{Lt_s} : \forall x.Lt(x,\mathbf{s}(x))}\,Lt_s}{\mathbf{Lt_s}\ \mathbf{0} : Lt(\mathbf{0},\mathbf{s}(\mathbf{0}))}\,\forall\mathsf{E}
}{(Lt_\neg\ \mathbf{0}\ \mathbf{s}(\mathbf{0}))\ (\mathbf{Eq_t}\ a\ \mathbf{0}\ \mathbf{s}(\mathbf{0})\ z)\ (\mathbf{Lt_s}\ \mathbf{0}) : \bot}\,\neg\mathsf{E}
}{\mathsf{let}\ \langle x,z\rangle = w\ \mathsf{in}\ (Lt_\neg\ \mathbf{0}\ \mathbf{s}(\mathbf{0}))\ (\mathbf{Eq_t}\ x\ \mathbf{0}\ \mathbf{s}(\mathbf{0})\ z)\ (\mathbf{Lt_s}\ \mathbf{0}) : \bot}\,\exists\mathsf{E}^a
}{\lambda w{:}\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathbf{s}(\mathbf{0})).\ \mathsf{let}\ \langle x,z\rangle = w\ \mathsf{in}\ (Lt_\neg\ \mathbf{0}\ \mathbf{s}(\mathbf{0}))\ (\mathbf{Eq_t}\ x\ \mathbf{0}\ \mathbf{s}(\mathbf{0})\ z)\ (\mathbf{Lt_s}\ \mathbf{0}) : \neg\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathbf{s}(\mathbf{0}))}\,\neg\mathsf{I}^w
$$

1.7 Sequent calculus

$$
\dfrac{\Gamma, \forall x.A, [t/x]A \longrightarrow C}{\Gamma, \forall x.A \longrightarrow C}\,\forall L
\qquad
\dfrac{\Gamma \longrightarrow [a/x]A}{\Gamma \longrightarrow \forall x.A}\,\forall R^a
$$

$$
\dfrac{\Gamma, \exists x.A, [a/x]A \longrightarrow C}{\Gamma, \exists x.A \longrightarrow C}\,\exists L^a
\qquad
\dfrac{\Gamma \longrightarrow [t/x]A}{\Gamma \longrightarrow \exists x.A}\,\exists R
$$

Todo. Proof of cut elimination.
Todo. Normalization

Corollary 1.1. *If* $\cdot \vdash \exists x.A$ *true, then there exists a term t such that* $\cdot \vdash [t/x]A$.

Examples

Axioms

In order to build a proof of $C\!\uparrow$ using the sequent calculus, we prove $\Gamma_{\mathsf{axiom}} \longrightarrow C$ where Γ_{axiom} contains all the axioms:

$$
\begin{array}{rcl}
\Gamma_{\mathsf{axiom}} & = & Nat(\mathbf{0}), \forall x.Nat(x) \supset Nat(\mathbf{s}(x)),\\
& & \forall x.Eq(x,x), \forall x.\forall y.\forall z.(Eq(x,y) \wedge Eq(x,z)) \supset Eq(y,z),\\
& & \forall x.Lt(x,\mathbf{s}(x)), \forall x.\forall y.Eq(x,y) \supset \neg Lt(x,y)
\end{array}
$$

Theorems

Proof of $\Gamma_{\mathsf{axiom}} \longrightarrow \forall x.\forall y.Eq(x,y) \supset Eq(y,x)$:

We let

$$
\begin{array}{rcl}
\Gamma'_{\mathsf{axiom}} & = & \Gamma_{\mathsf{axiom}},\\
& & \forall y.\forall z.(Eq(a,y) \wedge Eq(a,z)) \supset Eq(y,z),\\
& & \forall z.(Eq(a,b) \wedge Eq(a,z)) \supset Eq(b,z),\\
& & (Eq(a,b) \wedge Eq(a,a)) \supset Eq(b,a)
\end{array}
$$

$$
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\dfrac{}{\Gamma'_{\mathsf{axiom}}, Eq(a,b) \longrightarrow Eq(a,b)}\,Init
\quad
\dfrac{\dfrac{}{\Gamma'_{\mathsf{axiom}}, Eq(a,b), Eq(a,a) \longrightarrow Eq(a,a)}\,Init}{\Gamma'_{\mathsf{axiom}}, Eq(a,b) \longrightarrow Eq(a,a)}\,\forall L
}{\Gamma'_{\mathsf{axiom}}, Eq(a,b) \longrightarrow Eq(a,b) \wedge Eq(a,a)}\,\wedge R
\quad
\dfrac{}{\Gamma'_{\mathsf{axiom}}, Eq(a,b), Eq(b,a) \longrightarrow Eq(b,a)}\,Init
}{\Gamma'_{\mathsf{axiom}}, Eq(a,b) \longrightarrow Eq(b,a)}\,\supset L
}{\Gamma_{\mathsf{axiom}}, \forall y.\forall z.(Eq(a,y) \wedge Eq(a,z)) \supset Eq(y,z), \forall z.(Eq(a,b) \wedge Eq(a,z)) \supset Eq(b,z), Eq(a,b) \longrightarrow Eq(b,a)}\,\forall L
}{\Gamma_{\mathsf{axiom}}, \forall y.\forall z.(Eq(a,y) \wedge Eq(a,z)) \supset Eq(y,z), Eq(a,b) \longrightarrow Eq(b,a)}\,\forall L
}{\Gamma_{\mathsf{axiom}}, Eq(a,b) \longrightarrow Eq(b,a)}\,\forall L
}{\Gamma_{\mathsf{axiom}} \longrightarrow Eq(a,b) \supset Eq(b,a)}\,\supset R
}{\Gamma_{\mathsf{axiom}} \longrightarrow \forall y.Eq(a,y) \supset Eq(y,a)}\,\forall R^b
}{\Gamma_{\mathsf{axiom}} \longrightarrow \forall x.\forall y.Eq(x,y) \supset Eq(y,x)}\,\forall R^a
$$