Chapter 1

Propositional Logic

This chapter develops *propositional logic*, *i.e.*, logic without universal or existential quantifiers. We formulate propositional logic in the judgmental style of Pfenning and Davies [**?**], which adopts Martin-Löf's methodology of distinguishing between *propositions* and *judgments* [**?**]. It differs from the traditional style of formulating logic which relies solely on propositions.

1.1 Propositions and judgments

In a judgmental formulation of logic, a proposition is an object of verification whose *truth* can be checked by inference rules, whereas a judgment is an object of knowledge which becomes evident by a *proof*. Examples of propositions are *'1 + 1 is equal to 0'* and *'1 + 1 is equal to 2'*, both under inference rules based on arithmetic. Examples of judgments are *"'1 + 1 is equal to 0' is true"*, for which there is no proof, and *"'1 + 1 is equal to 2' is true,"* for which there is a proof.

To clarify the difference between propositions and judgments, consider a statement *'the moon is made of cheese.'* The statement is not yet an object of verification, or a proposition, since there is no way to check its truth — it becomes a proposition only when an inference rule is given. Here is an example of such an inference rule (written in a pedantic way):

$$\frac{\textit{`the moon is greenish white and has holes in it' is true}}{\textit{`the moon is made of cheese' is true}}\ \mathit{MoonCheese}$$

Now we can attempt to verify the proposition, for example, by taking a picture of the moon. That is, we still do not know whether the proposition is true or not, but by virtue of the inference rule, we know at least what counts as a verification of it. If the picture indeed shows that the moon is greenish white and has holes in it, the inference rule makes evident the judgment *"'the moon is made of cheese' is true."* Now we know *"'the moon is made of cheese' is true"* by the proof consisting of the picture and the inference rule. Thus a proposition is an object of verification which may or may not be true, whereas a judgment is an object of knowledge which we either know or do not know, depending on the existence of a proof.

It is important that the notion of judgment takes priority over the notion of proposition. Simply put, the notion of judgment does not depend on the notion of proposition, and we must introduce new kinds of judgments without using particular propositions. On the other hand, propositions are always explained with existing judgments, which include at least truth judgments because propositions must be accompanied by inference rules for establishing their truth.

In developing a formal system of propositional logic, we use two judgments: $A\ \mathit{prop}$ and $A\ \mathit{true}$.

$$\begin{array}{rcl} A\ \mathit{prop} & \Leftrightarrow & A \textit{ is a proposition} \\ A\ \mathit{true} & \Leftrightarrow & A \textit{ is true} \end{array}$$

$A\ prop$ becomes evident by the presence of an inference rule deducing $A\ true$. We will inductively define the set of propositions using binary connectives (*e.g.*, implication $\supset$, conjunction $\wedge$, disjunction $\vee$) and unary connectives (*e.g.*, negation $\neg$). The inference rules will be designed in such a way that the definition of a connective does not involve another connective. We say that the resultant system is *orthogonal* in the sense that all connectives can be developed independently of each other.

Exercise 1.1. Suppose that $\neg A$ is a proposition standing for the logical negation of A and that $A\ \mathit{false}$ is a falsehood judgment denoting *"A cannot be true."* What is wrong with the rule $\dfrac{\neg A\ true}{A\ \mathit{false}}\ \neg\mathsf{E}$ as a means of explaining the notion of falsehood judgments? What about $\dfrac{A\ \mathit{false}}{\neg A\ true}\ \neg\mathsf{I}$?

Exercise 1.2. Why is the rule $\dfrac{\neg A \vee B\ true}{A \supset B\ true}\ \supset\!\mathsf{I}$ bad, apart from its strange meaning?

1.2 Natural deduction system for propositional logic

Natural deduction [**?**] is a principle for building a system of logic whose main concepts are *introduction* and *elimination rules*. An introduction rule explains how to deduce a truth judgment involving a particular connective, exploiting those judgments in the premise. That is, it explains how to "introduce" the connective in a derivation (when read in the top-down way). For example, an introduction rule for the conjunction connective would look like:

$$\frac{\cdots}{A \wedge B\ true}\ \wedge\mathsf{I}$$

A dual concept is an elimination rule which explains how to exploit a truth judgment involving a particular connective to deduce another judgment in the conclusion. That is, it explains how to "eliminate" the connective in a derivation (when read in the top-down way). For example, an elimination rule for the conjunction connective would look like:

$$\frac{A \wedge B\ true}{\cdots}\ \wedge\mathsf{E}$$

An introduction rule usually conveys the intuition behind a connective and is thus relatively easy to design. In contrast, an elimination rule extracts the knowledge represented by a judgment and careful design is required to ensure that the resultant system is sound and complete in a sense to be explained in Section 1.5. For example, an ill-designed elimination rule may be so strong as to extract false knowledge that cannot be justified by its corresponding introduction rule. Or it may be too weak to deduce any interesting judgment. Note that an introduction rule takes precedence over its corresponding elimination rule because without an introduction rule, there is no use in designing an elimination rule. That is, an elimination rule cannot be considered separately from its corresponding introduction rule whereas the design of an introduction rule can be an isolated task.

Below we develop a natural deduction system for propositional logic, beginning with the conjunction connective $\wedge$ (which is the easiest case).

Conjunction

Before we investigate inference rules for $\wedge$, we need to know how to build valid propositions involving $\wedge$. Hence we need a *formation rule* to state that $A \wedge B$, read as *"A and B"* or *"A conjunction B,"* is a proposition if both A and B are propositions:

$$\frac{A\ prop \quad B\ prop}{A \wedge B\ prop}\ \wedge\mathsf{F}$$

In order to justify the rule $\wedge$F, we need an inference rule for proving the truth of $A \wedge B$ on the assumption that there are inference rules for proving the truth of A and B. Since $A \wedge B$ is intended to be true whenever both A and B are true, we use the following introduction rule to admit $A \wedge B$ as a proposition:

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \wedge B\ \mathit{true}}\ \wedge\mathsf{I}$$

The rule $\wedge\mathsf{I}$ says that if both A and B are true, then $A \wedge B$ is true. It follows the usual interpretation of an inference rule: if the premise holds, then the conclusion holds. Now we may use the rule $\wedge\mathsf{I}$ to construct a proof of $A \wedge B\ \mathit{true}$ from a proof $\mathcal{D}_A$ of $A\ \mathit{true}$ and a proof $\mathcal{D}_B$ of $B\ \mathit{true}$; we write $\begin{array}{c}\mathcal{D}_A \\ A\ \mathit{true}\end{array}$ to mean that $\mathcal{D}_A$ is a proof of $A\ \mathit{true}$, including the last inference rule whose conclusion is $A\ \mathit{true}$:

$$\frac{\begin{array}{c}\mathcal{D}_A \\ A\ \mathit{true}\end{array} \quad \begin{array}{c}\mathcal{D}_B \\ B\ \mathit{true}\end{array}}{A \wedge B\ \mathit{true}}\ \wedge\mathsf{I}$$

The design of an elimination rule for $\wedge$ begins with $A \wedge B\ \mathit{true}$ as a premise. Since $A \wedge B\ \mathit{true}$ expresses that both A and B are true, we may conclude either $A\ \mathit{true}$ or $B\ \mathit{true}$ from $A \wedge B\ \mathit{true}$, as shown in the two elimination rules for $\wedge$:

$$\frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}}\ \wedge\mathsf{E_L} \qquad \frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}}\ \wedge\mathsf{E_R}$$

Implication

The implication connective $\supset$ requires the notion of a *hypothetical proof* which is a proof containing *hypotheses*. We read $A \supset B$ as *"A implies B"* or *"if A, then B,"* and use the following formation rule:

$$\frac{A\ \mathit{prop} \quad B\ \mathit{prop}}{A \supset B\ \mathit{prop}}\ \supset\mathsf{F}$$

The intuition behind $\supset$ is that $A \supset B\ \mathit{true}$ holds whenever $A\ \mathit{true}$ implies $B\ \mathit{true}$, or a hypothesis of $A\ \mathit{true}$ leads to a proof of $B\ \mathit{true}$. We write a hypothesis of $A\ \mathit{true}$ as $\overline{A\ \mathit{true}}$, and obtain the following introduction rule for $\supset$:

$$\frac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x} \\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\ \supset\mathsf{I}^x$$

The premise of the rule $\supset\mathsf{I}^x$ is an example of a hypothetical proof because it contains a hypothesis, *i.e.*, a judgment that is assumed to hold. We say that the rule $\supset\mathsf{I}$ *internalizes* the hypothetical proof in its premise as a proposition $A \supset B$ in the sense that the truth of $A \supset B$ compactly represents the knowledge expressed by the hypothetical proof.

There are three observations to make about the rule $\supset\mathsf{I}^x$. First we annotate both the hypothesis $\overline{A\ \mathit{true}}$ and the rule name $\supset\mathsf{I}$ with the same label x. Thus a label in a hypothesis indicates from which inference rule the hypothesis originates. It is not necessary to annotate all hypotheses with different labels as long as no conflict occurs between two hypotheses with the same label. For example, the following derivation is okay even though both hypotheses are annotated with the same label x:

$$\frac{\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x} \\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\ \supset\mathsf{I}^x \quad \dfrac{\begin{array}{c}\overline{A'\ \mathit{true}}^{\,x} \\ \vdots \\ B'\ \mathit{true}\end{array}}{A' \supset B'\ \mathit{true}}\ \supset\mathsf{I}^x}{(A \supset B) \wedge (A' \supset B')\ \mathit{true}}$$

Second the hypothesis $\overline{A\ \mathit{true}}^{\,x}$ remains in effect only within the premise of the rule $\supset\mathsf{I}^x$. In other words, its scope is restricted to the premise of the rule $\supset\mathsf{I}^x$. After the rule $\supset\mathsf{I}^x$ is applied to deduce $A \supset B\ \mathit{true}$, $\overline{A\ \mathit{true}}^{\,x}$ may no longer be used as a valid hypothesis. We say that a hypothesis is *discharged* when its corresponding inference rule is applied and its scope is exited.

Note that while the premise of the rule $\supset\mathsf{I}^x$ is a hypothetical proof, the whole proof itself is *not* a hypothetical proof. Specifically the proof $\mathcal{D}$ below is a hypothetical proof, but the proof $\mathcal{E}$ is not:

$$\mathcal{E}\left\{\begin{array}{c} \mathcal{D}\left\{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \vdots \\ B\ \mathit{true}\end{array}\right. \\ \hline A \supset B\ \mathit{true}\end{array}\right. \supset\mathsf{I}^x$$

The reason why $\mathcal{E}$ is not a hypothetical proof is that the hypothesis $\overline{A\ \mathit{true}}^{\,x}$ is discharged when the rule $\supset\mathsf{I}^x$ is applied, and thus is not visible to the outside. That is, we are free to use any hypothesis without turning the whole proof into a hypothetical proof as long as it is eventually discharged.

Third the hypothesis $\overline{A\ \mathit{true}}^{\,x}$ may be used not just once but as many times as necessary. In fact, we may even ignore it in the proof without using it at all. Here are examples of proofs that ignore $\overline{A\ \mathit{true}}^{\,x}$, use it once, and use it twice:

$$\dfrac{\dfrac{\overline{B\ \mathit{true}}^{\,y} \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \textit{(not used in the proof)}\end{array}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^x}{B \supset (A \supset B)\ \mathit{true}}\supset\mathsf{I}^y \qquad \dfrac{\overline{A\ \mathit{true}}^{\,x}}{A \supset A\ \mathit{true}}\supset\mathsf{I}^x \qquad \dfrac{\dfrac{\overline{A\ \mathit{true}}^{\,x} \quad \overline{A\ \mathit{true}}^{\,x}}{A \wedge A\ \mathit{true}}\wedge\mathsf{I}}{A \supset (A \wedge A)\ \mathit{true}}\supset\mathsf{I}^x$$

As with the elimination rules for $\wedge$, the design of the elimination rule for $\supset$ begins with a premise $A \supset B\ \mathit{true}$. Since $A \supset B\ \mathit{true}$ expresses that $A\ \mathit{true}$ implies $B\ \mathit{true}$, the only way to exploit it is by supplying a proof of $A\ \mathit{true}$ to conclude $B\ \mathit{true}$. Hence the elimination rule for $\supset$ uses both $A \supset B\ \mathit{true}$ and $A\ \mathit{true}$ as its premises:

$$\dfrac{A \supset B\ \mathit{true} \quad A\ \mathit{true}}{B\ \mathit{true}}\supset\mathsf{E}$$

The following example proves $(A \supset B) \supset (A \supset B)\ \mathit{true}$ using the rule $\supset\mathsf{E}$:

$$\dfrac{\dfrac{\dfrac{\overline{A \supset B\ \mathit{true}}^{\,x} \quad \overline{A\ \mathit{true}}^{\,y}}{B\ \mathit{true}}\supset\mathsf{E}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^y}{(A \supset B) \supset (A \supset B)\ \mathit{true}}\supset\mathsf{I}^x$$

Disjunction

Like $\wedge$ and $\supset$, the disjunction connective $\vee$ is binary:

$$\dfrac{A\ \mathit{prop} \quad B\ \mathit{prop}}{A \vee B\ \mathit{prop}}\vee\mathsf{F}$$

$A \vee B$, read as "*A or B*" or "*A disjunction B*," is intended to be true when either A or B is true, but we do not necessarily know which alternative is true. In our formulation of propositional logic, an introduction rule for $\vee$ concludes $A \vee B\ \mathit{true}$ from a proof of either $A\ \mathit{true}$ or $B\ \mathit{true}$:

$$\dfrac{A\ \mathit{true}}{A \vee B\ \mathit{true}}\vee\mathsf{I}_\mathsf{L} \qquad \dfrac{B\ \mathit{true}}{A \vee B\ \mathit{true}}\vee\mathsf{I}_\mathsf{R}$$

The design of an elimination rule for $\vee$ is not obvious. A naive attempt would be to conclude one of $A\ true$ and $B\ true$ from $A \vee B\ true$:

$$\frac{A \vee B\ true}{A\ true}\ \vee\mathsf{E_L}? \qquad \frac{A \vee B\ true}{B\ true}\ \vee\mathsf{E_R}?$$

In a certain sense, both rules are too strong (or too powerful) because they conclude a judgment that cannot be justified by $A \vee B\ true$, which does not specify exactly which of $A\ true$ and $B\ true$ holds. In fact, each rule allows us to prove $A\ true$ for any proposition A:

$$\dfrac{\dfrac{\dfrac{\overline{B\ true}^{\,x}}{B \supset B\ true}\ \supset\mathsf{I}^x}{A \vee (B \supset B)\ true}\ \vee\mathsf{I_R}}{A\ true}\ \vee\mathsf{E_L}?$$

Since it is generally unknown which of $A\ true$ and $B\ true$ has been supplied in a proof of $A \vee B\ true$ (*e.g.*, when $A \vee B\ true$ is a hypothesis), the only logical way to exploit $A \vee B\ true$ is by considering both possibilities simultaneously. If we can prove $C\ true$ both from $A\ true$ and from $B\ true$ for a certain proposition C, then we may conclude $C\ true$ from $A \vee B\ true$, since $C\ true$ holds regardless of how the proof of $A \vee B\ true$ has been built. The elimination rule for $\vee$ expresses such a way of reasoning:

$$\dfrac{A \vee B\ true \qquad \begin{array}{c}\overline{A\ true}^{\,x}\\ \vdots \\ C\ true\end{array} \qquad \begin{array}{c}\overline{B\ true}^{\,y}\\ \vdots \\ C\ true\end{array}}{C\ true}\ \vee\mathsf{E}^{x,y}$$

Note that $A\ true$ and $B\ true$ are introduced as new hypotheses and are annotated with different labels x and y. As in the elimination rule for $\supset$, their scope is limited to their respective premises of the rule $\vee\mathsf{E}^{x,y}$ (*i.e.*, $\overline{A\ true}^{\,x}$ to the second premise and $\overline{B\ true}^{\,y}$ to the third premise), which means that both hypotheses are discharged when $C\ true$ is deduced in the conclusion.

Unlike the elimination rules for $\wedge$ and $\supset$, the elimination rule for $\vee$ exploits $A \vee B\ true$ in an indirect way in that its conclusion contains a proposition C that is not necessarily A, B, or their combination. That is, when applying the elimination rule to $A \vee B\ true$, we ourselves have to choose a proposition C (which can be completely unrelated to A and B) such that $C\ true$ is provable both from $A\ true$ and from $B\ true$. For this reason, the inclusion of $\vee$ in a system of logic makes it hard to investigate metalogical properties of the system, as we will see later.

As a trivial example, let us prove that $A\ true$ is stronger than $A \vee B\ true$:

$$\dfrac{\dfrac{\overline{A\ true}^{\,x}}{A \vee B\ true}\ \vee\mathsf{I_L}}{A \supset (A \vee B)\ true}\ \supset\mathsf{I}^x$$

The converse does not hold, *i.e.*, $A \vee B\ true$ is strictly weaker than $A\ true$, because there is no way to prove $A\ true$ from $B\ true$ for arbitrary propositions A and B:

$$\dfrac{\dfrac{\overline{A \vee B\ true}^{\,x} \qquad \overline{A\ true}^{\,y} \qquad \begin{array}{c}\overline{B\ true}^{\,z}\\ \vdots \\ A\ true \quad \textit{(impossible)}\end{array}}{A\ true}\ \vee\mathsf{E}^{y,z}}{(A \vee B) \supset A\ true}\ \supset\mathsf{I}^x$$

As another example, let us prove that the disjunction connective is commutative:

$$(A \vee B) \supset (B \vee A)\ true$$

We begin by applying the rule $\supset$I so that the problem reduces to proving $B \vee A$ *true* from $A \vee B$ *true*:

$$\dfrac{\dfrac{}{A \vee B\ \mathit{true}}\,x \quad \vdots \quad B \vee A\ \mathit{true}}{(A \vee B) \supset (B \vee A)\ \mathit{true}}\ \supset\mathsf{I}^x$$

At this point, the proof may proceed either in a bottom-up way by applying an introduction rule $\vee\mathsf{I_L}$ or $\vee\mathsf{I_R}$ to $B \vee A$ *true*, or in a top-down way by applying the elimination rule $\vee$E to $A \vee B$ *true*. In the first case, we eventually get stuck because it is impossible to prove A *true* or B *true* from $A \vee B$ *true*. For example, we cannot fill the gap in the proof shown below:

$$\dfrac{\dfrac{\dfrac{}{A \vee B\ \mathit{true}}\,x \quad \vdots \quad B\ \mathit{true}}{B \vee A\ \mathit{true}}\ \vee\mathsf{I_L}}{(A \vee B) \supset (B \vee A)\ \mathit{true}}\ \supset\mathsf{I}^x$$

In the second case, the problem reduces to separately proving $B \vee A$ *true* from A *true* and from B *true*, which is accomplished by applying the introduction rules for $\vee$:

$$\dfrac{\dfrac{\dfrac{}{A \vee B\ \mathit{true}}\,x \quad \dfrac{\dfrac{}{A\ \mathit{true}}\,y}{B \vee A\ \mathit{true}}\ \vee\mathsf{I_R} \quad \dfrac{\dfrac{}{B\ \mathit{true}}\,z}{B \vee A\ \mathit{true}}\ \vee\mathsf{I_L}}{B \vee A\ \mathit{true}}\ \vee\mathsf{E}^{y,z}}{(A \vee B) \supset (B \vee A)\ \mathit{true}}\ \supset\mathsf{I}^x$$

Exercise 1.3. We can rewrite the elimination rule for the disjunction connective by using the implication connective in place of hypothetical proofs:

$$\dfrac{A \vee B\ \mathit{true} \quad A \supset C\ \mathit{true} \quad B \supset C\ \mathit{true}}{C\ \mathit{true}}\ \vee\mathsf{E}$$

Why do we not use the new elimination rule which actually seems simpler than the previous one?

Truth and falsehood

Truth $\top$ is a proposition that is assumed to be always true. Hence a proof of $\top$ *true* requires no particular evidence and is always provable, as indicated by the empty premise in its introduction rule:

$$\dfrac{}{\top\ \mathit{prop}}\ \top\mathsf{F} \qquad \dfrac{}{\top\ \mathit{true}}\ \top\mathsf{I}$$

Then how do we exploit a proof of $\top$ *true* in an elimination rule? Since we have to provide no particular evidence in a proof of $\top$ *true*, there is no logical content in it, which implies that there is no interesting way to exploit it. Therefore $\top$ has no elimination rule.

Falsehood $\bot$ is a proposition that is never true, or equivalently, whose truth is impossible to establish. The intuition is that it denotes a logical contradiction which must not be provable under any circumstance. Therefore there is no introduction rule for $\bot$. Interestingly, however, there *is* an elimination rule for $\bot$. Suppose that we have a proof of $\bot$ *true*. If we think of $\bot$ *true* as something impossible to prove, or as something that is the most difficult to prove, the existence of its proof implies that we can prove everything (which is no more difficult to prove than $\bot$ *true*)! Therefore the elimination rule for $\bot$ deduces C *true* for an arbitrary proposition C:

$$\dfrac{}{\bot\ \mathit{prop}}\ \bot\mathsf{F} \qquad \dfrac{\bot\ \mathit{true}}{C\ \mathit{true}}\ \bot\mathsf{E}$$

Then why do we need an elimination rule for $\bot$ at all, if it is impossible to prove $\bot$ *true*? While it is impossible to prove $\bot$ *true* out of nothing, it is possible to prove $\bot$ *true* using hypotheses. For example, $\bot$ *true* in the premise of the rule $\bot\mathsf{E}$ itself may be a hypothesis, as illustrated in the proof below:

$$\dfrac{\dfrac{\overline{\bot\ \mathit{true}}^{\,x}}{C\ \mathit{true}}\ \bot\mathsf{E}}{\bot \supset C\ \mathit{true}}\ \supset\!\mathsf{I}^x$$

In essence, there is nothing wrong with reasoning from an assumption that something impossible to prove has been proven somehow.

We say that a system of logic is *inconsistent* if $\bot$ *true* is provable in it, and *consistent* if not. An inconsistent system is worthless because a judgment A *true* is provable for an arbitrary proposition A. We will later present a proof that our system of propositional logic is consistent, whose discovery was in fact a major milestone in the history of logic.

Truth $\top$ and falsehood $\bot$ can also be viewed as the nullary cases of conjunction and disjunction, respectively. Consider a general n-ary case $\bigwedge_{i=1}^n A_i$ of conjunction with a single introduction rule and n elimination rules:

$$\dfrac{A_i\ \mathit{true}\quad \text{for } i = 1, \cdots, n}{\bigwedge_{i=1}^n A_i\ \mathit{true}}\ \bigwedge\mathsf{I} \qquad \dfrac{\bigwedge_{i=1}^n A_i\ \mathit{true}}{A_i\ \mathit{true}}\ \bigwedge\mathsf{E}_i\ (1 \le i \le n)$$

If we let $\top = \bigwedge_{i=1}^n A_i$ with $n = 0$, the rule $\bigwedge\mathsf{I}$ turns into the rule $\top\mathsf{I}$ because it comes to have an empty premise, and each rule $\bigwedge\mathsf{E}_i$ disappears (*i.e.*, no elimination rule for $\top$). Similarly a general n-ary case $\bigvee_{i=1}^n A_i$ of disjunction has n introduction rules and a single elimination rule:

$$\dfrac{A_i\ \mathit{true}}{\bigvee_{i=1}^n A_i\ \mathit{true}}\ \bigvee\mathsf{I}_i (1 \le i \le n) \qquad \dfrac{\bigvee_{i=1}^n A_i\ \mathit{true} \qquad \begin{array}{c}\overline{A_i\ \mathit{true}}^{\,x_i}\\ \vdots \\ C\ \mathit{true}\end{array} \quad \text{for } i = 1, \cdots, n}{C\ \mathit{true}}\ \bigvee\mathsf{E}^x$$

If we let $\bot = \bigvee_{i=1}^n A_i$ with $n = 0$, each rule $\bigvee\mathsf{I}_i$ disappears (*i.e.*, no introduction rule for $\bot$), and the rule $\bigvee\mathsf{E}$ turns into the rule $\bot\mathsf{E}$ because all hypothetical proofs in its premise disappear.

Now it is clear that $\top$ and $\bot$ are identities for the binary connectives $\wedge$ and $\vee$, respectively. For example, we can identify $A \wedge \top$ with A: if A *true* is provable, then $A \wedge \top$ *true* is also provable because $\top$ *true* automatically holds; the converse follows by the rule $\wedge\mathsf{E_L}$. Similarly we can identify $A \vee \bot$ with A: if $A \vee \bot$ *true* is provable, A *true* must also be provable because the second alternative $\bot$ *true* cannot be taken; the converse follows by the rule $\vee\mathsf{I_L}$.

Negation

The only unary connective in propositional logic is negation $\neg$:

$$\dfrac{A\ \mathit{prop}}{\neg A\ \mathit{prop}}\ \neg\mathsf{F}$$

$\neg A$, read as "*not* A" or "*negation* A," denotes the logical negation of A, and its truth means that A cannot be true. Below we consider three different approaches to designing inference rules for negation, all of which provide a means to express that A cannot be true.

The first approach is to define a falsehood judgment A *false* denoting "A *cannot be true*" and then use the following rules to deduce and exploit $\neg A$ *true*:

$$\dfrac{A\ \mathit{false}}{\neg A\ \mathit{true}}\ \neg\mathsf{I} \qquad \dfrac{\neg A\ \mathit{true}}{A\ \mathit{false}}\ \neg\mathsf{E}$$

(We do not discuss inference rules for deducing $A\ \mathit{false}$.) As in the rule $\supset$I, we say that the rule $\neg$I internalizes $A\ \mathit{false}$ as a proposition $\neg A$ in the sense that the truth of $\neg A$ compactly represents the knowledge expressed by $A\ \mathit{false}$.

In the second approach, we deduce $\neg A\ \mathit{true}$ if an assumption of $A\ \mathit{true}$ leads to the provability of every truth judgment. The rationale is that if the system is known to be consistent (and thus not every truth judgment is provable), the provability of every truth judgment, *i.e.*, inconsistency of the system, as a consequence of an assumption of $A\ \mathit{true}$ implies that the assumption must be wrong, that is, A cannot be true.

In order to be able to express the provability of every truth judgment, we introduce a *propositional variable* p which stands for *any* proposition. We use a *parametric judgment* $p\ \mathit{true}$, or a judgment parametric in a propositional variable p, in the introduction rule for $\neg$:

$$\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \vdots \\ p\ \mathit{true}\end{array}}{\neg A\ \mathit{true}}\ \neg\mathsf{I}^{x,p}$$

Since the premise is a hypothetical judgment, we annotate the hypothesis $\overline{A\ \mathit{true}}$ and the rule name $\neg$I with the same label x. Moreover we annotate the rule name $\neg$I with the propositional variable p as well, since p is a fresh variable whose scope is restricted to the premise. The elimination rule for $\neg$ states that proofs of both $\neg A\ \mathit{true}$ and $A\ \mathit{true}$ license us to prove the truth of any proposition:

$$\dfrac{\neg A\ \mathit{true} \quad A\ \mathit{true}}{C\ \mathit{true}}\ \neg\mathsf{E}$$

Note that C in the conclusion can be any proposition, including propositional variables. As an example, we prove that A and $\neg A$ cannot be true simultaneously:

$$\dfrac{\dfrac{\dfrac{\overline{A \wedge \neg A\ \mathit{true}}^{\,x}}{\neg A\ \mathit{true}}\ \wedge\mathsf{E_R} \quad \dfrac{\overline{A \wedge \neg A\ \mathit{true}}^{\,x}}{A\ \mathit{true}}\ \wedge\mathsf{E_L}}{p\ \mathit{true}}\ \neg\mathsf{E}}{\neg(A \wedge \neg A)\ \mathit{true}}\ \neg\mathsf{I}^{x,p}$$

The third approach uses a *notational definition* by regarding $\neg A$ as a syntactic abbreviation of $A \supset \bot$. That is, $\neg$ plays no semantic role at all and $\neg A$ is simply expanded to $A \supset \bot$. The notational definition of $\neg$ justifies the following rules:

$$\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \vdots \\ \bot\ \mathit{true}\end{array}}{\neg A\ \mathit{true}}\ \neg\mathsf{I}^{x} \qquad \dfrac{\neg A\ \mathit{true} \quad A\ \mathit{true}}{\bot\ \mathit{true}}\ \neg\mathsf{E}$$

Note that if $\neg$ was defined as an independent connective rather than a notational convenience, these rules would destroy the orthogonality of the system because the meaning of $\neg$ would depend on the meaning of $\bot$. We use the third approach in our treatment of $\neg$ (which is the most popular definition in the literature).

As an example, we prove that if A is true, then $\neg A$ cannot be true:

$$\dfrac{\dfrac{\dfrac{\overline{\neg A\ \mathit{true}}^{\,y} \quad \overline{A\ \mathit{true}}^{\,x}}{\bot\ \mathit{true}}\ \neg\mathsf{E}}{\neg\neg A\ \mathit{true}}\ \neg\mathsf{I}^{y}}{A \supset \neg\neg A\ \mathit{true}}\ \supset\mathsf{I}^{x}$$

The converse $\neg\neg A \supset A\ \mathit{true}$ is *not* provable, however, which implies that $A\ \mathit{true}$ is strictly stronger than $\neg\neg A\ \mathit{true}$. That is, a proof that $\neg A$ cannot be true is not enough for concluding that A is true. A failed

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \wedge B\ \mathit{true}}\wedge\mathsf{I} \qquad \frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}}\wedge\mathsf{E_L} \qquad \frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}}\wedge\mathsf{E_R}$$

$$\frac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\supset\mathsf{I}^x \qquad \frac{A \supset B\ \mathit{true} \quad A\ \mathit{true}}{B\ \mathit{true}}\supset\mathsf{E}$$

$$\frac{A\ \mathit{true}}{A \vee B\ \mathit{true}}\vee\mathsf{I_L} \qquad \frac{B\ \mathit{true}}{A \vee B\ \mathit{true}}\vee\mathsf{I_R} \qquad \frac{A \vee B\ \mathit{true} \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \vdots \\ C\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,y}\\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\vee\mathsf{E}^{x,y}$$

$$\frac{}{\top\ \mathit{true}}\top\mathsf{I} \qquad \frac{\bot\ \mathit{true}}{C\ \mathit{true}}\bot\mathsf{E} \qquad \frac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \vdots \\ \bot\ \mathit{true}\end{array}}{\neg A\ \mathit{true}}\neg\mathsf{I}^x \qquad \frac{\neg A\ \mathit{true} \quad A\ \mathit{true}}{\bot\ \mathit{true}}\neg\mathsf{E}$$

Figure 1.1: Natural deduction system for propositional logic

attempt to prove $\neg\neg A \supset A\ \mathit{true}$ would look like:

$$\dfrac{\dfrac{\dfrac{\overline{\neg\neg A\ \mathit{true}}^{\,x} \qquad \begin{array}{c}\overline{\neg\neg A\ \mathit{true}}^{\,x}\\ \vdots\ ?\\ \neg A\ \mathit{true}\end{array}}{\bot\ \mathit{true}}\neg\mathsf{E}}{A\ \mathit{true}}\bot\mathsf{E}}{\neg\neg A \supset A\ \mathit{true}}\supset\mathsf{I}^x$$

The unprovability of $\neg\neg A \supset A\ \mathit{true}$ is a quintessential feature of the system of logic presented so far, or any system belonging to what is known as *constructive logic* or *intuitionistic logic*. In constructive logic, what $\neg A\ \mathit{true}$ proves is not exactly the direct opposite of what $A\ \mathit{true}$ proves. Rather it provides only indirect evidence that there is no proof of $A\ \mathit{true}$ by showing that the existence of such a proof leads to a logical contradiction. In contrast, *classical logic* assumes that every proposition is either true or false and has no intermediate state. Under classical logic, $\neg\neg A\ \mathit{true}$ is indistinguishable from $A\ \mathit{true}$ because A is either true or false and we have positive evidence that A cannot be false. The truth table method for proving the truth of a proposition is based on classical logic, which tries all possible combinations of truth and falsehood values for all atomic propositions. Until we come back to the topic of classical logic in Chapter **??**, we focus only on constructive logic.

Figure 1.1 shows all inference rules of propositional logic where the set of propositions is inductively defined as follows:

$$\text{proposition} \quad A \quad ::= \quad P \mid A \wedge A \mid A \supset A \mid A \vee A \mid \top \mid \bot \mid \neg A$$

P is called a *propositional constant* and denotes an atomic proposition (*e.g. '1 + 1 is equal to 0,' '1 + 1 is equal to 2' is true,' 'the moon is made of cheese,' etc*). The rules $\neg\mathsf{I}$ and $\neg\mathsf{E}$ are derived rules under the notational definition $\neg A = A \supset \bot$. From now on, we use the following operator precedence

$$\neg > \wedge > \vee > \supset$$

where $\wedge$, $\vee$, $\supset$ are all right-associative. Examples are:

$$\begin{array}{rcl} \neg A \wedge B & = & (\neg A) \wedge B \\ A \wedge B \vee C & = & (A \wedge B) \vee C \\ A \vee B \supset C & = & (A \vee B) \supset C \\ \neg A \wedge B \vee C \supset D & = & (((\neg A) \wedge B) \vee C) \supset D \end{array} \qquad \begin{array}{rcl} A \wedge B \wedge C & = & A \wedge (B \wedge C) \\ A \vee B \vee C & = & A \vee (B \vee C) \\ A \supset B \supset C & = & A \supset (B \supset C) \end{array}$$

1.3 Logical equivalence

We say that a proposition A is logically equivalent to another proposition B, written $A \equiv B$, if A *true* implies B *true* and vice versa. A notational definition of logical equivalence $A \equiv B$ is given as follows:

$$A \equiv B \quad = \quad (A \supset B) \wedge (B \supset A)\ true$$

If A and B are logically equivalent, an occurrence of A inside any proposition may be replaced by B (or an occurrence of B by A) without changing its meaning in that the resultant proposition remains logically equivalent to the original proposition. Thus logical equivalences enable us to simplify a proof involving a proposition that is logically equivalent to a less complex proposition. For example,

$$\neg\neg\neg A \supset (\neg\neg\neg B \supset \neg(A \vee B))\ true$$

becomes easy (or even obvious) to prove once we transform $\neg\neg\neg A \supset (\neg\neg\neg B \supset \neg(A \vee B))$ into $(\neg A \wedge \neg B) \supset \neg(A \vee B)$ by exploiting logical equivalences $\neg\neg\neg A \equiv \neg A$ and $A \supset (B \supset C) \equiv (A \wedge B) \supset C$.

Below we list logical equivalences of propositional logic which are divided into three groups.

Commutativity and idempotence. $\wedge$ and $\vee$ are commutative and idempotent. An implication $A \supset A$ is logically meaningless and reduces to $\top$.

(C1) $A \wedge B \equiv B \wedge A$
(C2) $A \vee B \equiv B \vee A$
(C3) $A \supset B \not\equiv B \supset A$
(I1) $A \wedge A \equiv A$
(I2) $A \vee A \equiv A$
(I3) $A \supset A \equiv \top$

Truth and falsehood. Each logical equivalence below deals with a proposition of the form $\top\ \phi\ A$, $\bot\ \phi\ A$, $A \supset \top$, or $A \supset \bot$ where ϕ is $\wedge$, $\vee$, or $\supset$.

(M1) $\top \wedge A \equiv A$
(M2) $\top \vee A \equiv \top$
(M3) $\top \supset A \equiv A$
(M4) $\bot \wedge A \equiv \bot$
(M5) $\bot \vee A \equiv A$
(M6) $\bot \supset A \equiv \top$
(M7) $A \supset \top \equiv \top$
(M8) $A \supset \bot = \neg A$

Interaction between connectives. Each logical equivalence below deals with a proposition of the form $A\ \phi\ (B\ \phi\ C)$ or $(A\ \phi\ B) \supset C$ where ϕ is $\wedge$, $\vee$, or $\supset$.

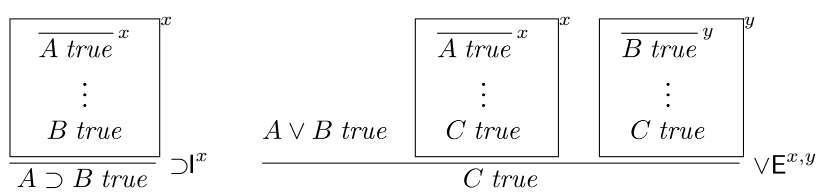
(L1)	$A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$	*(associativity of* $\wedge$*)*
(L2)	$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$	*(distributivity of* $\wedge$ *over* $\vee$*)*
(L3)	$A \wedge (B \supset C) \equiv ?$	*(no interaction)*
(L4)	$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$	*(distributivity of* $\vee$ *over* $\wedge$*)*
(L5)	$A \vee (B \vee C) \equiv (A \vee B) \vee C$	*(associativity of* $\vee$*)*
(L6)	$A \vee (B \supset C) \equiv ?$	*(no interaction)*
(L7)	$A \supset (B \wedge C) \equiv (A \supset B) \wedge (A \supset C)$	*(distributivity of* $\supset$ *over* $\wedge$*)*
(L8)	$A \supset (B \vee C) \equiv ?$	*(no interaction)*
(L9)	$A \supset (B \supset C) \equiv (A \wedge B) \supset C$	
(L10)	$(A \wedge B) \supset C \equiv A \supset (B \supset C)$	
(L11)	$(A \vee B) \supset C \equiv (A \supset C) \wedge (B \supset C)$	
(L12)	$(A \supset B) \supset C \equiv ?$	*(no interaction)*

1.4 Hypothetical judgments

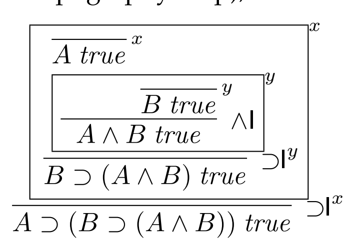
While the rules in Figure 1.1 define a natural deduction system for propositional logic, they are unwieldy for writing hypothetical proofs. This is because no rule provides visual aid for keeping track of the scope of each hypothesis or an apparatus for preventing a hypothesis from escaping its scope. For example, the following hypothetical proof contains a wrong use of a hypothesis $\overline{A\ \mathit{true}}^{x}$ outside its scope:

$$\dfrac{\dfrac{\overline{A\ \mathit{true}}^{x}}{A \supset A\ \mathit{true}}\supset\mathsf{I}^{x} \quad \overline{A\ \mathit{true}}^{x}\mathit{(wrong\ use)}}{(A \supset A) \wedge A\ \mathit{true}}\wedge\mathsf{I}$$

Whenever a new hypothesis is introduced, therefore, we could draw an imaginary contour to delineate its scope, as illustrated below:



Here the scope of a hypothesis $\overline{A\ \mathit{true}}^{x}$ is restricted to the contour labeled x. Note that a contour may be enclosed within another contour (as in a topography map), as shown in the following example:



The hypothesis $\overline{A\ \mathit{true}}^{x}$ may be used inside the inner contour labeled y (which makes it possible to prove $A \wedge B\ \mathit{true}$ by the rule $\wedge\mathsf{I}$), but the hypothesis $\overline{B\ \mathit{true}}^{y}$ cannot be used outside the inner contour.

Hypothetical judgments provide a convenient way to keep track of the scope of each hypothesis in a hypothetical proof. A hypothetical judgment $J_1, \cdots, J_n \vdash J$ becomes evident by a hypothetical proof deducing a judgment J from a collection of hypotheses $\overline{J_1}, \cdots, \overline{J_n}$:

$$J_1, \cdots, J_n \vdash J \quad \Longleftrightarrow \quad \begin{array}{ccc} \overline{J_1} & \cdots & \overline{J_n} \\ & \ddots\ \therefore\ \cdot^{\cdot^{\cdot}} & \\ & J & \end{array} \quad \} \text{ inference rules}$$

Thus we may read $J_1, \cdots, J_n \vdash J$ as "if judgments $J_1, \cdots, J_n$ hold, then a judgment J hold." When $J_1, \cdots, J_n \vdash J$ holds, we say that J_1 through J_n *entail* J, or J is a consequence of J_1 through J_n. Hence $\vdash$ is called an *entailment relation* or a *consequence relation*. We refer to J_i, $1 \leq i \leq n$, as an *antecedent* and J as the *succedent*. We often abbreviate a collection of antecedents as Γ, as in a hypothetical judgment $\Gamma \vdash J$.

In developing a natural deduction system for propositional logic, we will use hypothetical judgments of the form $A_1\ \mathit{true}, \cdots, A_n\ \mathit{true} \vdash A\ \mathit{true}$ where antecedents and succedents are all truth judgments. Before presenting its inference rules, let us investigate properties of hypothetical judgments of the general form where antecedents and conclusions can be any judgments.

The definition of hypothetical judgments justifies two principles: *reflexivity* and *substitution principle*:

- (Reflexivity) $\Gamma, J, \Gamma' \vdash J$.
- (Substitution principle) If $\Gamma \vdash J$ and $\Gamma, J \vdash J'$, then $\Gamma \vdash J'$.

Reflexivity states that we may use a hypothesis $\overline{J}$ to conclude J. The substitution principle states that if we can prove a judgment J from a collection of hypotheses, we may use J as another hypothesis whenever the same collection of hypotheses is available. That is, we may use J as a lemma once we build a proof of $\Gamma \vdash J$.

To see how the substitution principle works, let us assume $\Gamma \vdash J$ and $\Gamma, J \vdash J'$ which imply that there are two hypothetical proofs $\mathcal{D}$ and $\mathcal{E}$ as shown below:

$$\Gamma \vdash J \quad \Longleftrightarrow \quad \left.\begin{array}{c} \overline{\Gamma} \\ \vdots \\ J \end{array}\right\} \mathcal{D} \qquad\qquad \Gamma, J \vdash J' \quad \Longleftrightarrow \quad \left.\begin{array}{ccc} \overline{\Gamma} & \cdots & \overline{J} \\ & \ddots\ \cdot^{\cdot^{\cdot}} & \\ & J' & \end{array}\right\} \mathcal{E}$$

Here $\overline{\Gamma}$ is a shorthand for $\{\overline{J} \mid J \in \Gamma\}$. Now we locate every occurrence of the hypothesis $\overline{J}$ in $\mathcal{E}$ and *substitute* $\mathcal{D}$ for it, which results in the following hypothetical proof:

$$\begin{array}{ccc} & & \left.\begin{array}{c} \overline{\Gamma} \\ \vdots \\ J \end{array}\right\} \mathcal{D} \\ \overline{\Gamma} & \cdots & \\ & \ddots\ \cdot^{\cdot^{\cdot}} & \\ & J' & \end{array}$$

Since the same hypothesis (*e.g.*, one in $\overline{\Gamma}$) may be used as many times as necessary (see Page 4), the hypothetical judgment above makes evident the hypothetical judgment $\Gamma \vdash J'$, which is what the substitution principle concludes from $\Gamma \vdash J$ and $\Gamma, J \vdash J'$.

Our definition of hypothetical judgments makes two implicit assumptions: 1) the order of hypotheses is immaterial; 2) a hypothesis may be used zero or more times in a hypothetical proof. These assumptions are formally stated in the *structural properties* of hypothetical judgments:

- (Exchange) If $\Gamma, J_i, J_{i+1}, \Gamma' \vdash J$, then $\Gamma, J_{i+1}, J_i, \Gamma' \vdash J$.
- (Weakening) If $\Gamma, \Gamma' \vdash J$, then $\Gamma, J', \Gamma' \vdash J$ for any judgment J'.
- (Contraction) If $\Gamma, J_i, J_i, \Gamma' \vdash J$, then $\Gamma, J_i, \Gamma' \vdash J$.

Exchange states that we may ignore the order of antecedents in a hypothetical judgment. Weakening states that we may add a new antecedent without using it, thereby "weakening" what is being proven. ($\Gamma, J' \vdash J$ is weaker than $\Gamma \vdash J$ because it draws the same conclusion from more hypotheses.) By contraction, we may combine two copies of the same antecedent into one. Note that by weakening, a hypothesis may be used zero times and that by contraction, a hypothesis may be used more than once.

Here are a few further remarks on hypothetical judgments:

- Hypothetical judgments are just a *"convenient"* way, rather than a new way, to represent hypothetical proofs. That is, the entailment relation $\vdash$ is just a syntactic tool for displaying the hypotheses and the conclusion of a hypothetical judgment while hiding its internal structure, and thus does *not* introduce a new semantic notion. (In contrast, the relation $\models$ from model theory defines the notion of *semantic* consequence. Hence it has nothing to do with hypothetical judgments and is *not* a syntactic convenience.)

- There can be more than one hypothetical proof by which a given hypothetical judgment becomes evident, since hypothetical judgments is concerned only with hypotheses and conclusions.

- A hypothetical judgment itself is an example of a judgment and thus may be used as an antecedent or the conclusion in another hypothetical judgment, although we will not use hypothetical judgments in such a way in our discussion of logic. In fact, $J_1, \cdots, J_n \vdash J$ can be thought of as an abbreviation of a nested hypothetical judgment $J_1 \vdash (J_2 \vdash \cdots (J_n \vdash J) \cdots)$, where each antecedent J_i or the conclusion J may be another hypothetical judgment!

- While closely related to each other, a hypothetical judgment $J_1, \cdots, J_n \vdash J$ and a rule $\dfrac{J_1 \quad \cdots \quad J_n}{J}\; R$ are disparate concepts and thus impossible to compare for equivalence. The reason is simple: the former is a judgment whereas the latter is an inference rule. The existence of a proof of $J_1, \cdots, J_n \vdash J$ just implies that R is a derivable rule. Conversely, if the rule R is available, we can always prove $J_1, \cdots, J_n \vdash J$ with the following hypothetical proof:

$$\dfrac{\overline{J_1} \quad \cdots \quad \overline{J_n}}{J}\; R$$

  Note, however, that the above hypothetical proof may not be the only way to prove $J_1, \cdots, J_n \vdash J$ if we can build another hypothetical proof $\begin{array}{c}\overline{J_1} \quad \cdots \quad \overline{J_n} \\ \ddots \quad \therefore \\ J\end{array}$ without using the rule R at all.

- A hypothetical judgment $\cdot \vdash J$ with no antecedents is *not* equivalent to its succedent J. While the former states that J holds unconditionally (or categorically), the latter is unaware of whether there are hypotheses or not, and could be even a hypothesis in a hypothetical judgment. For example, from the assumption that J entails J' (*i.e.*, $J \vdash J'$), we can show that $\cdot \vdash J$ implies $\cdot \vdash J'$ by the substitution principle. The converse is not the case, however, because a proof of $\cdot \vdash J'$ does not necessarily extend a proof of $\cdot \vdash J$ so that J follows directly from J'. (If $\cdot \vdash J$ and J were equivalent, the converse would also be the case.)

- An important consequence of the structural properties is that the two hypothetical judgments in each rule, $\Gamma \vdash J$ from the *if* part and $\Gamma' \vdash J$ from the *then* part, represent hypothetical proofs not only of the same size (in terms of the number of applications of inference rules) but also of completely the same structure. As a result, when structural induction (or rule induction) is applicable to $\Gamma \vdash J$, *we may apply structural induction on* $\Gamma' \vdash J$ *instead.*

Figure 1.2 shows a natural deduction system for propositional logic using hypothetical judgments, which reuses the inference rule names from the previous natural deduction system. We use hypothetical judgments of the form $\Gamma \vdash A\ true$ where Γ is a collection of truth judgments and the exchange rule is built-in (*i.e.*, we may reorder antecedents as we like).

The rule Hyp expresses reflexivity of hypothetical judgments. All the other rules are justified by their counterparts in the previous natural deduction system. As an example, let us consider the rule $\supset$I. The premise $\Gamma, A\ true \vdash B\ true$ implies the existence of a hypothetical proof deducing $B\ true$ from hypotheses

$$
\frac{A\ \mathit{true} \in \Gamma}{\Gamma \vdash A\ \mathit{true}}\ \mathsf{Hyp} \qquad
\frac{\Gamma, A\ \mathit{true} \vdash B\ \mathit{true}}{\Gamma \vdash A \supset B\ \mathit{true}}\ {\supset}\mathsf{I} \qquad
\frac{\Gamma \vdash A \supset B\ \mathit{true} \quad \Gamma \vdash A\ \mathit{true}}{\Gamma \vdash B\ \mathit{true}}\ {\supset}\mathsf{E}
$$

$$
\frac{\Gamma \vdash A\ \mathit{true} \quad \Gamma \vdash B\ \mathit{true}}{\Gamma \vdash A \wedge B\ \mathit{true}}\ {\wedge}\mathsf{I} \qquad
\frac{\Gamma \vdash A \wedge B\ \mathit{true}}{\Gamma \vdash A\ \mathit{true}}\ {\wedge}\mathsf{E_L} \qquad
\frac{\Gamma \vdash A \wedge B\ \mathit{true}}{\Gamma \vdash B\ \mathit{true}}\ {\wedge}\mathsf{E_R}
$$

$$
\frac{\Gamma \vdash A\ \mathit{true}}{\Gamma \vdash A \vee B\ \mathit{true}}\ {\vee}\mathsf{I_L} \qquad
\frac{\Gamma \vdash B\ \mathit{true}}{\Gamma \vdash A \vee B\ \mathit{true}}\ {\vee}\mathsf{I_R} \qquad
\frac{\Gamma \vdash A \vee B\ \mathit{true} \quad \Gamma, A\ \mathit{true} \vdash C\ \mathit{true} \quad \Gamma, B\ \mathit{true} \vdash C\ \mathit{true}}{\Gamma \vdash C\ \mathit{true}}\ {\vee}\mathsf{E}
$$

$$
\frac{}{\Gamma \vdash \top\ \mathit{true}}\ {\top}\mathsf{I} \qquad
\frac{\Gamma \vdash \bot\ \mathit{true}}{\Gamma \vdash C\ \mathit{true}}\ {\bot}\mathsf{E} \qquad
\frac{\Gamma, A\ \mathit{true} \vdash \bot}{\Gamma \vdash \neg A\ \mathit{true}}\ {\neg}\mathsf{I} \qquad
\frac{\Gamma \vdash \neg A\ \mathit{true} \quad \Gamma \vdash A\ \mathit{true}}{\Gamma \vdash \bot\ \mathit{true}}\ {\neg}\mathsf{E}
$$

Figure 1.2: Natural deduction system using hypothetical judgments

$\overline{\Gamma}$ and $\overline{A\ \mathit{true}}$ (where $\overline{\Gamma}$ is a shorthand for $\{\overline{J}\,|J \in \Gamma\}$):

$$
\Gamma, A\ \mathit{true} \vdash B\ \mathit{true} \quad \Longleftrightarrow \quad
\begin{array}{ccc}
\overline{\Gamma} & \cdots & \overline{A\ \mathit{true}} \\
 & \ddots\ \therefore & \\
 & B\ \mathit{true} &
\end{array}
$$

Now we apply the rule $\supset$I (in Figure 1.1) to the conclusion $B\ \mathit{true}$ *with respect to the hypothesis* $\overline{A\ \mathit{true}}$. That is, the application of the rule $\supset$I designates $\overline{A\ \mathit{true}}$ as its corresponding hypothesis:

$$
\frac{\begin{array}{ccc}
\overline{\Gamma} & \cdots & \overline{A\ \mathit{true}}^{\,x} \\
 & \ddots\ \therefore & \\
 & B\ \mathit{true} &
\end{array}}{A \supset B\ \mathit{true}}\ {\supset}\mathsf{I}^x
$$

Note that hypotheses in the proof include hypotheses $\overline{\Gamma}$ but *not* $\overline{A\ \mathit{true}}^{\,x}$, which is discharged when the rule $\supset$I is applied. That is, we have a hypothetical proof for the hypothetical judgment $\Gamma \vdash A \supset B$:

$$
\frac{\begin{array}{ccc}
\overline{\Gamma} & \cdots & \overline{A\ \mathit{true}}^{\,x} \\
 & \ddots\ \therefore & \\
 & B\ \mathit{true} &
\end{array}}{A \supset B\ \mathit{true}}\ {\supset}\mathsf{I}^x
\quad \Longleftrightarrow \quad \Gamma \vdash A \supset B\ \mathit{true}
$$

Thus we can prove $\Gamma \vdash A \supset B\ \mathit{true}$ whenever we have a proof of $\Gamma, A\ \mathit{true} \vdash B\ \mathit{true}$, which justifies the rule $\supset$I in Figure 1.2.

The use of hypothetical judgments eliminates the need to annotate hypotheses with labels. For example, here is a proof of a hypothetical judgment $\cdot \vdash A \supset (B \supset (A \wedge B))\ \mathit{true}$:

$$
\cfrac{\cfrac{\cfrac{\cfrac{}{A\ \mathit{true}, B\ \mathit{true} \vdash A\ \mathit{true}}\ \mathsf{Hyp} \quad \cfrac{}{A\ \mathit{true}, B\ \mathit{true} \vdash B\ \mathit{true}}\ \mathsf{Hyp}}{A\ \mathit{true}, B\ \mathit{true} \vdash A \wedge B\ \mathit{true}}\ {\wedge}\mathsf{I}}{A\ \mathit{true} \vdash B \supset (A \wedge B)\ \mathit{true}}\ {\supset}\mathsf{I}}{\cdot \vdash A \supset (B \supset (A \wedge B))\ \mathit{true}}\ {\supset}\mathsf{I}
$$

There are two observations to make about the new natural deduction system. First each leaf in a derivation tree for $\Gamma \vdash C\ \mathit{true}$ (where $\Gamma \vdash C\ \mathit{true}$ is regarded as the root) is an application of either the rule Hyp or the rule $\top$I. In particular, if the rule $\top$I is not used (which is often the case), the derivation tree has the following form:

$$\dfrac{}{\Gamma_1 \vdash A_1\ \mathit{true}}\ \mathsf{Hyp} \qquad \cdots \qquad \dfrac{}{\Gamma_n \vdash A_n\ \mathit{true}}\ \mathsf{Hyp}$$
$$\ddots\ \cdot\ \cdot^{\cdot}$$
$$\Gamma \vdash C\ \mathit{true}$$

Second the set of antecedents always expands in an inference rule as we move from the conclusion to its premises (*i.e.*, in a bottom-up way). That is, an inference rule $\dfrac{\Gamma' \vdash A\ \mathit{true} \quad \cdots}{\Gamma \vdash C\ \mathit{true}}$ satisfies $\Gamma \subset \Gamma'$. Then the above derivation tree for $\Gamma \vdash C\ \mathit{true}$ satisfies $\Gamma \subset \Gamma_1, \cdots, \Gamma \subset \Gamma_n$.

Weakening and contraction are now stated as follows:

Proposition 1.4 (Structural properties).
(Weakening) *If* $\Gamma \vdash C\ \mathit{true}$*, then* $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$.
(Contraction) *If* $\Gamma, A\ \mathit{true}, A\ \mathit{true} \vdash C\ \mathit{true}$*, then* $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$.

Proof. By induction on the structure of the proof of $\Gamma \vdash C\ \mathit{true}$ and $\Gamma, A\ \mathit{true}, A\ \mathit{true} \vdash C\ \mathit{true}$. For weakening, the proof of $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$ has exactly the same structure as the proof of $\Gamma \vdash C\ \mathit{true}$. When structural induction is applicable to $\Gamma \vdash C\ \mathit{true}$, therefore, we may apply structural induction on $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$ instead. A similar observation holds for contraction. □

The provability of the substitution principle confirms that the system in Figure 1.2 adheres to the definition of hypothetical judgments. That is, if the substitution principle was unprovable, it would indicate that some rule in Figure 1.2 was not designed according to the relation between hypothetical judgments and hypothetical proofs.

Theorem 1.5 (Substitution). *If* $\Gamma \vdash A\ \mathit{true}$ *and* $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$*, then* $\Gamma \vdash C\ \mathit{true}$.

Exercise 1.6. To which judgment do you think structural induction must be applied in the proof of Theorem 1.5? $\Gamma \vdash A\ \mathit{true}$ or $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$? Why?

Before attempting to write a proof of Theorem 1.5, it is worthwhile to predict how the proof would proceed. It helps us, for example, to determine to which of $\Gamma \vdash A\ \mathit{true}$ and $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$ structural induction must be applied. For the sake of simplicity, let us assume that the rule $\top\mathsf{I}$ is not used in the proof of $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$. (The proof of $\Gamma \vdash A\ \mathit{true}$ may use the rule $\top\mathsf{I}$.) Then the derivation tree for $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$ has the following form

$$\dfrac{}{\Gamma_1, A\ \mathit{true} \vdash C_1\ \mathit{true}}\ \mathsf{Hyp} \qquad \cdots \qquad \dfrac{}{\Gamma_n, A\ \mathit{true} \vdash C_n\ \mathit{true}}\ \mathsf{Hyp}$$
$$\ddots\ \cdot\ \cdot^{\cdot}$$
$$\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$$

where each leaf $\dfrac{}{\Gamma_i, A\ \mathit{true} \vdash C_i\ \mathit{true}}\ \mathsf{Hyp}$ satisfies $\Gamma \subset \Gamma_i$ for $1 \leq i \leq n$. Now consider the i-the leaf. If $C_i\ \mathit{true} \in \Gamma_i$, the antecedent $A\ \mathit{true}$ in $\Gamma_i, A\ \mathit{true} \vdash C_i\ \mathit{true}$ plays no role in the proof and the leaf is safely replaced by $\dfrac{}{\Gamma_i \vdash C_i\ \mathit{true}}\ \mathsf{Hyp}$. If $C_i = A$, we weaken $\Gamma \vdash A\ \mathit{true} = \Gamma \vdash C_i\ \mathit{true}$ to obtain $\Gamma_i \vdash C_i\ \mathit{true}$, which is then *substituted* for $\Gamma_i, A\ \mathit{true} \vdash C_i\ \mathit{true}$. Now no leaf contains $A\ \mathit{true}$ as an antecedent, and by propagating these changes all the way down to the root, we transform the whole derivation tree into a new one for $\Gamma \vdash C\ \mathit{true}$. Thus we analyze the structure of the proof of $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$ to located all leaves in it. That is, we apply structural induction on $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$ rather than $\Gamma \vdash A\ \mathit{true}$.

Proof. By induction on the structure of the proof of $\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}$.

We consider three cases Hyp, $\supset$I, and $\supset$E.

Case $\dfrac{C\ \mathit{true} \in \Gamma}{\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}}\ \mathsf{Hyp}$

$\Gamma \vdash C\ \mathit{true}$ by the rule Hyp with $C\ \mathit{true} \in \Gamma$

Case $\dfrac{}{\Gamma, A\ true \vdash C\ true}$ Hyp where $A = C$

$\Gamma \vdash C\ true$ from the assumption $\Gamma \vdash A\ true$

Case $\dfrac{\Gamma, A\ true, C_1\ true \vdash C_2\ true}{\Gamma, A\ true \vdash C_1 \supset C_2\ true}$ $\supset$I where $C = C_1 \supset C_2$

$\Gamma, C_1\ true \vdash A\ true$ by weakening $\Gamma \vdash A\ true$

$\Gamma, C_1\ true \vdash C_2\ true$ by IH on $\Gamma, A\ true, C_1\ true \vdash C_2\ true$ with $\Gamma, C_1\ true \vdash A\ true$

$\Gamma \vdash C_1 \supset C_2\ true$ by the rule $\supset$I with $\Gamma, C_1\ true \vdash C_2\ true$

Case $\dfrac{\Gamma, A\ true \vdash C' \supset C\ true \quad \Gamma, A\ true \vdash C'\ true}{\Gamma, A\ true \vdash C\ true}$ $\supset$E

$\Gamma \vdash C' \supset C\ true$ by IH on $\Gamma, A\ true \vdash C' \supset C\ true$ with $\Gamma \vdash A\ true$

$\Gamma \vdash C'\ true$ by IH on $\Gamma, A\ true \vdash C'\ true$ with $\Gamma \vdash A\ true$

$\Gamma \vdash C\ true$ by the rule $\supset$E with $\Gamma \vdash C' \supset C\ true$ and $\Gamma \vdash C'\ true$

□

Thus, given a proof $\mathcal{D}$ of $\Gamma \vdash A\ true$ and a proof $\mathcal{E}$ of $\Gamma, A\ true \vdash C\ true$, we can always produce a proof, written as $[\mathcal{D}/A\ true]\mathcal{E}$, of $\Gamma \vdash C\ true$ by substituting $\mathcal{D}$ into $\mathcal{E}$.

1.5 Local soundness and completeness

All the inference rules presented so far seem to make sense intuitively, but their correctness is yet to be established in a formal way. For example, we would certainly expect an elimination rule for $\wedge$ by which $A\ true$ is deducible from $A \wedge B\ true$, but not an elimination rule that deduces $C\ true$ from $A \wedge B\ true$ if C is unrelated to A and B. Then, in designing a natural deduction system, what is the guiding principle to which we can appeal in order to decide whether to accept or reject an inference rule? The answer is that the system must satisfy two properties: *local soundness* and *local completeness*.

An introduction rule compresses the knowledge expressed in its premises into a truth judgment in the conclusion, whereas an elimination rule retrieves the knowledge compressed within a truth judgment in a premise to deduce another judgment in the conclusion. The local soundness property states that the knowledge retrieved from a judgment by an elimination rule is only part of the knowledge compressed within that judgment. Therefore, if local soundness fails, the elimination rule is too strong in the sense that it is capable of contriving some knowledge that cannot be justified by that judgment; thus local soundness ensures that the elimination rule is not too strong. The local completeness property states that the knowledge retrieved from a judgment by an elimination rule includes at least the knowledge compressed within that judgment. Therefore, if local completeness fails, the elimination rule is too weak in the sense that it is incapable of retrieving all the knowledge compressed within that judgment; thus local completeness ensures that the elimination rule is strong enough. If an elimination rule satisfies both properties, it retrieves exactly the same knowledge compressed within a judgment in a premise.

We verify the local soundness property by showing how to reduce a proof in which an introduction rule is immediately followed by a corresponding elimination rule. As an example, consider the following proof in which the introduction rule $\wedge$I is immediately followed by its corresponding elimination rule $\wedge\mathsf{E_L}$:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{A\ true} \quad \overset{\mathcal{E}}{B\ true}}{A \wedge B\ true}\ \wedge\mathsf{I}}{A\ true}\ \wedge\mathsf{E_L}$$

The rule $\wedge\mathsf{E_L}$ is not too strong because what it deduces in the conclusion, namely $A\ true$, is one of the two judgments used to deduce $A \wedge B\ \mathit{true}$. Hence the whole proof reduces to a simpler proof $\mathcal{D}$:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{A\ \mathit{true}} \quad \overset{\mathcal{E}}{B\ \mathit{true}}}{A \wedge B\ \mathit{true}}\,\wedge\mathsf{I}}{A\ \mathit{true}}\,\wedge\mathsf{E_L} \qquad \Longrightarrow_R \qquad \overset{\mathcal{D}}{A\ \mathit{true}}$$

If the rule $\wedge\mathsf{E_L}$ was too strong (*e.g.*, deducing $A \supset B\ \mathit{true}$ somehow), the proof would not be reducible.

As another example, consider the following proof in which the introduction rule $\supset\mathsf{I}$ is immediately followed by the elimination rule $\supset\mathsf{E}$:

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\,\supset\mathsf{I}^x \quad \overset{\mathcal{D}}{A\ \mathit{true}}}{B\ \mathit{true}}\,\supset\mathsf{E}$$

The rule $\supset\mathsf{E}$ is not too strong because the whole proof reduces to a smaller proof of the same judgment $B\ \mathit{true}$ by substituting $\mathcal{D}$ for the hypothesis $\overline{A\ \mathit{true}}^{\,x}$ in the premise of the rule $\supset\mathsf{I}^x$:

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\,\supset\mathsf{I}^x \quad \overset{\mathcal{D}}{A\ \mathit{true}}}{B\ \mathit{true}}\,\supset\mathsf{E} \qquad \Longrightarrow_R \qquad \begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\\ \vdots \\ B\ \mathit{true}\end{array}$$

For the natural deduction system based on hypothetical judgments, the substitution principle justifies $\Gamma \vdash B\ \mathit{true}$ when proofs of $\Gamma \vdash A\ \mathit{true}$ and $\Gamma, A\ \mathit{true} \vdash B\ \mathit{true}$ are given:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, A\ \mathit{true} \vdash B\ \mathit{true}}}{\Gamma \vdash A \supset B\ \mathit{true}}\,\supset\mathsf{I} \quad \overset{\mathcal{E}}{\Gamma \vdash A\ \mathit{true}}}{\Gamma \vdash B\ \mathit{true}}\,\supset\mathsf{E} \qquad \Longrightarrow_R \qquad \begin{array}{c}[\mathcal{E}/A\ \mathit{true}]\mathcal{D}\\ \Gamma \vdash B\ \mathit{true}\end{array}$$

The case for $\vee$ is similar to the case for $\supset$ and uses the substitution principle:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{A\ \mathit{true}}}{A \vee B\ \mathit{true}}\,\vee\mathsf{I_L} \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \vdots \\ C\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,y}\\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\,\vee\mathsf{E}^{x,y} \qquad \Longrightarrow_R \qquad \begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\\ \vdots \\ C\ \mathit{true}\end{array}$$

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \vdash A\ \mathit{true}}}{\Gamma \vdash A \vee B\ \mathit{true}}\,\vee\mathsf{I_L} \quad \overset{\mathcal{E}_L}{\Gamma, A\ \mathit{true} \vdash C\ \mathit{true}} \quad \overset{\mathcal{E}_R}{\Gamma, B\ \mathit{true} \vdash C\ \mathit{true}}}{\Gamma \vdash C\ \mathit{true}}\,\vee\mathsf{E} \qquad \Longrightarrow_R \qquad \begin{array}{c}[\mathcal{D}/A\ \mathit{true}]\mathcal{E}_L\\ \Gamma \vdash C\ \mathit{true}\end{array}$$

We refer to these reductions $\Longrightarrow_R$ as *local reductions*. Note that there are no local reductions for $\top$ and $\bot$, since $\top$ has no elimination rule and $\bot$ has no introduction rule.

We verify the local completeness property by showing how to expand a proof of a judgment into another proof in which one or more elimination rules are followed by an introduction rule for the same judgment. As an example, consider a proof $\mathcal{D}$ of $A \wedge B\ \mathit{true}$. The elimination rules $\wedge\mathsf{E_L}$ and $\wedge\mathsf{E_R}$ are not too weak because what they deduce in their conclusions, namely $A\ \mathit{true}$ and $B\ \mathit{true}$, are sufficient to reconstruct another proof of $A \wedge B\ \mathit{true}$:

$$\begin{array}{c}\mathcal{D}\\ A\wedge B\ \mathit{true}\end{array} \quad\Longrightarrow_E\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\wedge B\ \mathit{true}\end{array}}{A\ \mathit{true}}\wedge\mathsf{E_L} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ A\wedge B\ \mathit{true}\end{array}}{B\ \mathit{true}}\wedge\mathsf{E_R}}{A\wedge B\ \mathit{true}}\wedge\mathsf{I}$$

If the elimination rules were too weak (*e.g.*, being unable to deduce $A\ \mathit{true}$ somehow), the proof would not be expandable.

As another example, consider a proof $\mathcal{D}$ of $A \supset B\ \mathit{true}$. We can reconstruct another proof of the same judgment after applying the elimination rule $\supset\mathsf{E}$ to $\mathcal{D}$, which implies that the rule $\supset\mathsf{E}$ is not too weak:

$$\begin{array}{c}\mathcal{D}\\ A\supset B\ \mathit{true}\end{array} \quad\Longrightarrow_E\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\supset B\ \mathit{true}\end{array} \qquad \overline{A\ \mathit{true}}^{\,x}}{B\ \mathit{true}}\supset\mathsf{E}}{A\supset B\ \mathit{true}}\supset\mathsf{I}^x$$

In expanding the proof $\mathcal{D}$, we have to choose a fresh label x that is not already in use in $\mathcal{D}$, for any undischarged hypothesis $\overline{B\ \mathit{true}}^{\,x}$ with the same label x in $\mathcal{D}$ becomes associated with the rule $\supset\mathsf{I}^x$, resulting in an incorrect proof if $A \neq B$. For the natural deduction system based on hypothetical judgments, we weaken a proof $\mathcal{D}$ of $\Gamma \vdash A \supset B\ \mathit{true}$ to obtain a proof of $\Gamma, A\ \mathit{true} \vdash A \supset B\ \mathit{true}$ when reconstructing another proof of $\Gamma \vdash A \supset B\ \mathit{true}$:

$$\begin{array}{c}\mathcal{D}\\ \Gamma\vdash A\supset B\ \mathit{true}\end{array} \quad\Longrightarrow_E\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, A\ \mathit{true}\vdash A\supset B\ \mathit{true}\end{array} \qquad \dfrac{}{\Gamma, A\ \mathit{true}\vdash A\ \mathit{true}}\mathsf{Hyp}}{\Gamma, A\ \mathit{true}\vdash B\ \mathit{true}}\supset\mathsf{E}}{\Gamma\vdash A\supset B\ \mathit{true}}\supset\mathsf{I}$$

The case for $\vee$ is given as follows:

$$\begin{array}{c}\mathcal{D}\\ A\vee B\ \mathit{true}\end{array} \quad\Longrightarrow_E\quad \dfrac{\begin{array}{c}\mathcal{D}\\ A\vee B\ \mathit{true}\end{array} \qquad \dfrac{\overline{A\ \mathit{true}}^{\,x}}{A\vee B\ \mathit{true}}\vee\mathsf{I_L} \qquad \dfrac{\overline{B\ \mathit{true}}^{\,y}}{A\vee B\ \mathit{true}}\vee\mathsf{I_R}}{A\vee B\ \mathit{true}}\vee\mathsf{E}^{x,y}$$

$$\begin{array}{c}\mathcal{D}\\ \Gamma\vdash A\vee B\ \mathit{true}\end{array} \quad\Longrightarrow_E\quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma\vdash A\vee B\ \mathit{true}\end{array} \qquad \dfrac{\dfrac{}{\Gamma, A\ \mathit{true}\vdash A\ \mathit{true}}\mathsf{Hyp}}{\Gamma, A\ \mathit{true}\vdash A\vee B\ \mathit{true}}\vee\mathsf{I_L} \qquad \dfrac{\dfrac{}{\Gamma, B\ \mathit{true}\vdash B\ \mathit{true}}\mathsf{Hyp}}{\Gamma, B\ \mathit{true}\vdash A\vee B\ \mathit{true}}\vee\mathsf{I_R}}{\Gamma\vdash A\vee B\ \mathit{true}}\vee\mathsf{E}$$

We refer to these expansions $\Longrightarrow_E$ as *local expansions*.

Although there are no local reductions for $\top$ and $\bot$, there *are* local expansions for $\top$ and $\bot$. Recall that $\top$ and $\bot$ are the nullary cases of conjunction and disjunction, respectively. Hence a proof $\mathcal{D}$ of $\top\ \mathit{true}$ expands to another proof of $\top\ \mathit{true}$ that uses zero elimination rules and thus ignores $\mathcal{D}$:

$$\begin{array}{c}\mathcal{D}\\ \top\ \mathit{true}\end{array} \quad\Longrightarrow_E\quad \dfrac{}{\top\ \mathit{true}}\top\mathsf{I}$$

Similarly a proof of $\bot\ \mathit{true}$ expands to another proof of $\bot\ \mathit{true}$ that uses zero introduction rules:

$$\begin{array}{c}\mathcal{D}\\ \bot\ \mathit{true}\end{array} \quad\Longrightarrow_E\quad \dfrac{\begin{array}{c}\mathcal{D}\\ \bot\ \mathit{true}\end{array}}{\bot\ \mathit{true}}\bot\mathsf{E}$$

As every connective satisfies local soundness and completeness, the natural deduction system for propositional logic is said to be locally sound and complete. When the system is extended with a new connective, quantifier, or modality, we have to check that the system remains locally sound and complete by finding its local reduction and expansion, as we will see later.

1.6 Normal proofs

We have seen that a proof containing a *detour*, *i.e.*, an introduction rule immediately followed by a corresponding elimination rule, can be transformed to another proof by applying a local reduction. It turns out that for every proof of $A\ true$, there is a sequence of local reductions that lead to another proof of $A\ true$ containing no detour (the normalization theorem). We refer to the resultant proof as a *normal proof*. A normal proof is the most direct proof because a detour may be thought of as an example of indirect reasoning. Moreover it is minimal in size in a certain sense, irrespective of the size of its syntactic representation, because it does not reduce to another proof.

Since a normal proof contains no detour, it has the following form where top-down applications of elimination rules meet bottom-up applications of introduction rules in the middle:

$$\dfrac{}{A_1\ true}x_1 \quad \ldots \quad \dfrac{}{A_n\ true}x_n$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

elimination rules

$$\downarrow \qquad \downarrow$$

$$\downarrow$$

$$\uparrow$$

$$\uparrow \qquad \uparrow$$

introduction rules

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$C\ true$$

Thus the structure of a normal proof conforms to our intuition in building a proof by repeatedly applying introduction rules in a bottom-up way, adding new hypotheses, and repeatedly applying elimination rules in a top-down way, starting from hypotheses. Here is an example of a normal proof of $(A \wedge B) \supset (B \wedge A)\ true$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B\ true}x}{B\ true}\wedge\mathsf{E_R} \quad \dfrac{\dfrac{}{A \wedge B\ true}x}{A\ true}\wedge\mathsf{E_L}}{B \wedge A\ true}\wedge\mathsf{I}}{(A \wedge B) \supset (B \wedge A)\ true}\supset\mathsf{I}^x$$

A proof of $(A \wedge B) \supset (B \wedge A)\ true$ that is not normal contains detours in it:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B\ true}x}{A\ true}\wedge\mathsf{E_L} \quad \dfrac{\dfrac{}{A \wedge B\ true}x}{B\ true}\wedge\mathsf{E_R}}{A \wedge B\ true}\wedge\mathsf{I}\ (detour)}{B\ true}\wedge\mathsf{E_R} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B\ true}x}{A\ true}\wedge\mathsf{E_L} \quad \dfrac{\dfrac{}{A \wedge B\ true}x}{B\ true}\wedge\mathsf{E_R}}{A \wedge B\ true}\wedge\mathsf{I}\ (detour)}{A\ true}\wedge\mathsf{E_L}}{B \wedge A\ true}\wedge\mathsf{I}}{(A \wedge B) \supset (B \wedge A)\ true}\supset\mathsf{I}^x$$

Normal proofs are an indispensable tool in the study of logic because of their soundness and completeness properties: $A\ true$ holds if and only if there is a normal proof of $A\ true$. The soundness property holds trivially because a normal proof is just a proof of a special form. The completeness property (that every proof has a corresponding normal proof) has two important consequences. First, in order to prove $A\ true$, it suffices to find a normal proof of it. When proving $A\ true$, for example, it is safe to ignore proofs of the

$$
\frac{\begin{array}{c}\overline{A\downarrow}^{\,x}\\ \vdots\\ B\uparrow\end{array}}{A \supset B\uparrow}\,{\supset}\mathsf{I}^{x}_{\uparrow}
\qquad
\frac{A \supset B\downarrow \quad A\uparrow}{B\downarrow}\,{\supset}\mathsf{E}_{\downarrow}
\qquad
\frac{A\uparrow \quad B\uparrow}{A \wedge B\uparrow}\,{\wedge}\mathsf{I}_{\uparrow}
\qquad
\frac{A \wedge B\downarrow}{A\downarrow}\,{\wedge}\mathsf{E}_{\mathsf{L}\downarrow}
\qquad
\frac{A \wedge B\downarrow}{B\downarrow}\,{\wedge}\mathsf{E}_{\mathsf{R}\downarrow}
$$

$$
\frac{A\uparrow}{A \vee B\uparrow}\,{\vee}\mathsf{I}_{\mathsf{L}\uparrow}
\qquad
\frac{B\uparrow}{A \vee B\uparrow}\,{\vee}\mathsf{I}_{\mathsf{R}\uparrow}
\qquad
\frac{A \vee B\downarrow \quad \begin{array}{c}\overline{A\downarrow}^{\,x}\\ \vdots\\ C\uparrow\end{array} \quad \begin{array}{c}\overline{B\downarrow}^{\,y}\\ \vdots\\ C\uparrow\end{array}}{C\uparrow}\,{\vee}\mathsf{E}^{x,y}_{\uparrow}
$$

$$
\frac{}{\top\uparrow}\,{\top}\mathsf{I}_{\uparrow}
\qquad
\frac{\bot\downarrow}{C\uparrow}\,{\bot}\mathsf{E}_{\uparrow}
\qquad
\frac{A\downarrow}{A\uparrow}\,{\downarrow\!\uparrow}
$$

Figure 1.3: Inference rules for neutral and normal judgments

following form which cannot be normal proofs:

$$
\frac{\begin{array}{c}\vdots\\ B \supset A\ \mathit{true}\end{array} \qquad \begin{array}{c}\vdots\\ B\ \mathit{true}\end{array}}{A\ \mathit{true}}\,{\supset}\mathsf{E}
$$

That is, we need to concern ourselves only with the most direct proof rather an indirect proof, for example, by introducing an intermediate proposition B as shown above. Second, in order to refute $A\ \mathit{true}$, it suffices to try to build a normal proof of it (by alternating between bottom-up applications of introduction rules and top-down applications of elimination rules) and show that the process gets stuck or does not terminate.

Exercise 1.7. Give an informal argument why $\neg\neg A \supset A\ \mathit{true}$ is not provable.

To formalize all these ideas, we introduce two new judgments: *neutral judgments* $A\downarrow$ and *normal judgments* $A\uparrow$. A neutral judgment $A\downarrow$ becomes evident by a *neutral proof* of $A\ \mathit{true}$ which is either a hypothesis or an elimination rule applied to another neutral proof, whereas a normal judgment $A\uparrow$ becomes evident by a normal proof of $A\ \mathit{true}$ which is either a neutral judgment or an introduction rule applied to another normal proof. Thus the direction of the arrow in each judgment coincides with the direction in which the proof construction should proceed. Specifically we exploit an existing neutral judgment $A\downarrow$ in order to deduce another judgment by determining which elimination rule to be applied; hence the proof construction from a neutral judgment always proceeds downward (↓). For a normal judgment $A\uparrow$ whose proof is incomplete yet, we determine which introduction rule must be applied in order to deduce it; hence the proof construction from a normal judgment always proceeds upward (↑). When a neutral judgment $A\downarrow$ meets a normal judgment $A\uparrow$ in the middle, the proof construction is finished.

Figure 1.3 shows the inference rules for neutral and normal judgments. The rule ⇃↾, called the *coercion rule*, says that a neutral proof is a normal proof, and a typical construction of a normal proof is completed with an application of the rule ⇃↾. All the other rules are designed according to our intuition on building the most direct proofs, and thus are best read by following the direction of the arrow in each judgment. For example, suppose that we wish to build (↑) a new proof of $A \supset B\ \mathit{true}$:

$$
\begin{array}{c}\vdots\\ A \supset B\uparrow\end{array}
$$

In order to build the most direct proof of $A \supset B\ \mathit{true}$, we first assume $A\ \mathit{true}$ as a hypothesis which is to be exploited (↓) in deducing another judgment:

$$\begin{array}{c}\overline{A\!\downarrow}\\ \vdots\\ A \supset B\!\uparrow\end{array}$$

Then we try to build (↑) a proof of B *true*, which is precisely what the rule $\supset\mathsf{I}_\uparrow$ expresses:

$$\dfrac{\begin{array}{c}\overline{A\!\downarrow}\\ \vdots\\ B\!\uparrow\end{array}}{A \supset B\!\uparrow}$$

As another example, suppose that we wish to exploit (↓) an existing proof of $A \supset B$ *true*:

$$\begin{array}{c}A \supset B\!\downarrow\\ \vdots\end{array}$$

In order to exploit it in the most direct manner, we need a proof of A *true*, which we do not have yet. Therefore we first build (↑) a new proof of A *true*:

$$\begin{array}{c}A \supset B\!\downarrow \quad A\!\uparrow\\ \vdots\end{array}$$

A proof of A *true* then allows us to deduce B *true*. Since we now have a proof of B *true* ready for use in deducing another judgment, we classify it as a neutral judgment, which is precisely what the rule $\supset\mathsf{E}_\downarrow$ expresses:

$$\dfrac{A \supset B\!\downarrow \quad A\!\uparrow}{B\!\downarrow}$$

Exercise 1.8. Analyze all the remaining rules in an analogous way. Note that the rule $\vee\mathsf{E}_\uparrow$ superficially deduces a normal judgment $C\!\uparrow$ by applying an elimination rule, thereby contradicting our intuition on a normal judgment which is supposed to be either a neutral judgment or an introduction rule applied to another normal judgment. The essence of the proof of $C\!\uparrow$, however, is found not in the application of the rule $\vee\mathsf{E}_\uparrow$ itself but in the two premises deducing $C\!\uparrow$. In this regard, the rule $\vee\mathsf{E}_\uparrow$ still adheres to our intuition on normal judgments. The rules $\top\mathsf{I}_\uparrow$ and $\bot\mathsf{E}_\uparrow$ are obtained as the nullary cases of $\wedge\mathsf{I}_\uparrow$ and $\vee\mathsf{E}_\uparrow$, respectively.

The rules in Figure 1.3 are all designed in such a way that a proof of a neutral or normal judgment contains no detour. First observe that no proof of a neutral judgment ends with an application of an introduction rule (see the rules $\supset\mathsf{E}_\downarrow$, $\wedge\mathsf{E}_{\mathsf{L}\downarrow}$, $\wedge\mathsf{E}_{\mathsf{R}\downarrow}$). Then observe that the principal premise in each elimination rule is a neutral judgment (*e.g.*, $A \supset B\!\downarrow$ in the rule $\supset\mathsf{E}_\downarrow$), which has been shown not to end with an introduction rule and thus does not give rise to a detour.

As an example, we show that the proof of $(A \wedge B) \supset (B \wedge A)$ *true* given earlier is indeed a normal proof by rewriting it in terms of neutral and normal judgments; we annotate each judgment in it with either ↓ or ↑ according to the rules in Figure 1.3 and check that no conflicting annotation arises:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \wedge B\!\downarrow}^{\,x}}{B\!\downarrow}\wedge\mathsf{E}_{\mathsf{R}\downarrow}}{B\!\uparrow}\downarrow\!\uparrow \quad \dfrac{\dfrac{\overline{A \wedge B\!\downarrow}^{\,x}}{A\!\downarrow}\wedge\mathsf{E}_{\mathsf{L}\downarrow}}{A\!\uparrow}\downarrow\!\uparrow}{B \wedge A\!\uparrow}\wedge\mathsf{I}_\uparrow}{(A \wedge B) \supset (B \wedge A)\!\uparrow}\supset\mathsf{I}^x_\uparrow$$

$$
\frac{}{\Gamma_\downarrow, A{\downarrow} \vdash A{\downarrow}}\ \mathsf{Hyp}_\downarrow
\qquad
\frac{\Gamma_\downarrow, A{\downarrow} \vdash B{\uparrow}}{\Gamma_\downarrow \vdash A \supset B{\uparrow}}\ {\supset}\mathsf{I}_\uparrow
\qquad
\frac{\Gamma_\downarrow \vdash A \supset B{\downarrow} \quad \Gamma_\downarrow \vdash A{\uparrow}}{\Gamma_\downarrow \vdash B{\downarrow}}\ {\supset}\mathsf{E}_\downarrow
$$

$$
\frac{\Gamma_\downarrow \vdash A{\uparrow} \quad \Gamma_\downarrow \vdash B{\uparrow}}{\Gamma_\downarrow \vdash A \wedge B{\uparrow}}\ {\wedge}\mathsf{I}_\uparrow
\qquad
\frac{\Gamma_\downarrow \vdash A \wedge B{\downarrow}}{\Gamma_\downarrow \vdash A{\downarrow}}\ {\wedge}\mathsf{E}_{\mathsf{L}\downarrow}
\qquad
\frac{\Gamma_\downarrow \vdash A \wedge B{\downarrow}}{\Gamma_\downarrow \vdash B{\downarrow}}\ {\wedge}\mathsf{E}_{\mathsf{R}\downarrow}
$$

$$
\frac{\Gamma_\downarrow \vdash A{\uparrow}}{\Gamma_\downarrow \vdash A \vee B{\uparrow}}\ {\vee}\mathsf{I}_{\mathsf{L}\uparrow}
\qquad
\frac{\Gamma_\downarrow \vdash B{\uparrow}}{\Gamma_\downarrow \vdash A \vee B{\uparrow}}\ {\vee}\mathsf{I}_{\mathsf{R}\uparrow}
\qquad
\frac{\Gamma_\downarrow \vdash A \vee B{\downarrow} \quad \Gamma_\downarrow, A{\downarrow} \vdash C{\uparrow} \quad \Gamma_\downarrow, B{\downarrow} \vdash C{\uparrow}}{\Gamma_\downarrow \vdash C{\uparrow}}\ {\vee}\mathsf{E}_\uparrow
$$

$$
\frac{}{\Gamma_\downarrow \vdash \top{\uparrow}}\ {\top}\mathsf{I}_\uparrow
\qquad
\frac{\Gamma_\downarrow \vdash \bot{\downarrow}}{\Gamma_\downarrow \vdash C{\uparrow}}\ {\bot}\mathsf{E}_\uparrow
\qquad
\frac{\Gamma_\downarrow \vdash A{\downarrow}}{\Gamma_\downarrow \vdash A{\uparrow}}\ {\downarrow}{\uparrow}
$$

Figure 1.4: Inference rules for neutral and normal judgments using hypothetical judgments

Note that a detour is impossible to annotate with arrows $\downarrow$ and $\uparrow$:

$$
\frac{\dfrac{A{\uparrow} \quad B{\uparrow}}{A \wedge B \ {\uparrow}?\ {\downarrow}?}\ {\wedge}\mathsf{I}_\uparrow}{A{\downarrow}}\ {\wedge}\mathsf{E}_{\mathsf{L}\downarrow}
\qquad
\frac{\dfrac{\begin{matrix}\overline{A{\downarrow}}^{\,x} \\ \vdots \\ B{\uparrow}\end{matrix}}{A \supset B \ {\uparrow}?\ {\downarrow}?}\ {\supset}\mathsf{I}^x_\uparrow \quad A{\uparrow}}{B{\downarrow}}\ {\supset}\mathsf{E}_\downarrow
\qquad
\frac{\dfrac{A{\uparrow}}{A \vee B \ {\uparrow}?\ {\downarrow}?}\ {\vee}\mathsf{I}_{\mathsf{L}\uparrow} \quad \begin{matrix}\overline{A{\downarrow}}^{\,x} \\ \vdots \\ C{\uparrow}\end{matrix} \quad \begin{matrix}\overline{B{\downarrow}}^{\,y} \\ \vdots \\ C{\uparrow}\end{matrix}}{C{\uparrow}}\ {\vee}\mathsf{E}^{x,y}_\downarrow
$$

As another example, we show that no proof of $A \vee \neg A{\uparrow}$ exists where A is an arbitrary proposition:

$$
\frac{\begin{matrix}\textit{(stuck)} \\ A{\uparrow}\end{matrix}}{A \vee \neg A{\uparrow}}\ {\vee}\mathsf{I}_{\mathsf{L}\uparrow}
\qquad
\frac{\dfrac{\begin{matrix}\overline{A{\downarrow}}^{\,x} \\ \textit{(stuck)} \\ \vdots \\ \textit{(stuck)} \\ \bot{\uparrow}\end{matrix}}{\neg A{\uparrow}}\ {\supset}\mathsf{I}^x_\uparrow}{A \vee \neg A{\uparrow}}\ {\vee}\mathsf{I}_{\mathsf{R}\uparrow}
$$

(Hence $A \vee \neg A\ \mathit{true}$ is not provable in constructive logic, although it is a tautology in classical logic.)

Figure 1.4 shows an equivalent system for neutral and normal judgments using hypothetical judgments $\Gamma_\downarrow \vdash A{\uparrow}$ and $\Gamma_\downarrow \vdash A{\downarrow}$, where $\Gamma_\downarrow = \{A{\downarrow} \mid A \in \Gamma\}$ is a collection of neutral judgments and the exchange rule is built-in; we reuse the inference rule names from the previous system for neutral and normal judgments. The two structural properties, weakening and contraction, are stated as expected. As it is based on hypothetical judgments, the system also satisfies the substitution principle.

Proposition 1.9 (Structural properties).

(Weakening)	*If* $\Gamma_\downarrow \vdash C{\downarrow}$, *then* $\Gamma_\downarrow, A{\downarrow} \vdash C{\downarrow}$.
	If $\Gamma_\downarrow \vdash C{\uparrow}$, *then* $\Gamma_\downarrow, A{\downarrow} \vdash C{\uparrow}$.
(Contraction)	*If* $\Gamma_\downarrow, A{\downarrow}, A{\downarrow} \vdash C{\downarrow}$, *then* $\Gamma_\downarrow, A{\downarrow} \vdash C{\downarrow}$.
	If $\Gamma_\downarrow, A{\downarrow}, A{\downarrow} \vdash C{\uparrow}$, *then* $\Gamma_\downarrow, A{\downarrow} \vdash C{\uparrow}$.

Theorem 1.10 (Substitution).
If $\Gamma_\downarrow \vdash A{\downarrow}$ *and* $\Gamma_\downarrow, A{\downarrow} \vdash C{\downarrow}$, *then* $\Gamma_\downarrow \vdash C{\downarrow}$.
If $\Gamma_\downarrow \vdash A{\downarrow}$ *and* $\Gamma_\downarrow, A{\downarrow} \vdash C{\uparrow}$, *then* $\Gamma_\downarrow \vdash C{\uparrow}$.

Proof. By induction on the structure of the proof of $\Gamma_\downarrow, A{\downarrow} \vdash C{\downarrow}$ and $\Gamma_\downarrow, A{\downarrow} \vdash C{\uparrow}$. □

1.7 Normalization

We state the soundness and completeness properties of normal proofs using normal judgments as follows:

Theorem 1.11.
(Soundness) *If $A\!\uparrow$ is provable, then $A\ \mathit{true}$ is provable.*
(Completeness) *If $A\ \mathit{true}$ is provable, then $A\!\uparrow$ is provable.*

The soundness property is easy to show because every rule in Figure 1.3 is transformed into a corresponding rule in Figure 1.1 by replacing all neutral and normal judgments by truth judgments (*i.e.*, $A\!\uparrow$ and $A\!\downarrow$ by $A\ \mathit{true}$). In conjunction with the fact that a proof of a neutral or normal judgment contains no detour, it implies that $A\!\uparrow$ expresses a particular strategy for proving $A\ \mathit{true}$, namely a strategy that disallows detours. For the completeness property, we prove the normalization theorem shown below. For the moment, we do not consider disjunction $\lor$ and falsehood $\bot$. (Chapter **??** gives another formal proof covering all connectives in propositional logic.)

Theorem 1.12 (Normalization). *For every proof of $A\ \mathit{true}$, there is a sequence of local reductions that lead to a proof of $A\!\uparrow$. That is, the proof of $A\ \mathit{true}$ resulting from the sequence of local reductions can be annotated with arrows $\downarrow$ and $\uparrow$ according to the rules in Figure 1.3.*

Another theorem, called the *strong normalization theorem*, states that every sequence of local reductions, *regardless of their order*, eventually terminates to yield a normal proof:

Theorem 1.13 (Strong normalization). *Every sequence of local reductions starting from a proof of $A\ \mathit{true}$ terminates with a proof of $A\!\uparrow$. That is, there is no infinite sequence of local reductions starting from any proof of $A\ \mathit{true}$.*

In the presence of disjunction $\lor$ and falsehood $\bot$ (*i.e.*, in the full system of propositional logic), however, the normalization theorem fails! That is, a proof to which no local reduction can be applied is *not* necessarily a normal proof. (The converse that a normal proof contains no detour still holds.) For example, the following proof of $(A \lor A) \supset A\ \mathit{true}$ is not normal, but contains no detour, either:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \lor A\ \mathit{true}}\,z \quad \dfrac{\dfrac{}{A\ \mathit{true}}\,x \quad \dfrac{}{A\ \mathit{true}}\,x}{A \land A\ \mathit{true}}\,\land\mathsf{I} \quad \dfrac{\dfrac{}{A\ \mathit{true}}\,y \quad \dfrac{}{A\ \mathit{true}}\,y}{A \land A\ \mathit{true}}\,\land\mathsf{I}}{A \land A\ \mathit{true}}\,\lor\mathsf{E}^{x,y}}{A\ \mathit{true}}\,\land\mathsf{E_L}}{(A \lor A) \supset A\ \mathit{true}}\,\supset\mathsf{I}^z$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \lor A\!\downarrow}\,z \quad \dfrac{\dfrac{}{A\!\downarrow\uparrow}\,x \quad \dfrac{}{A\!\downarrow\uparrow}\,x}{A \land A\!\uparrow}\,\land\mathsf{I}_\uparrow \quad \dfrac{\dfrac{}{A\!\downarrow\uparrow}\,y \quad \dfrac{}{A\!\downarrow\uparrow}\,y}{A \land A\!\uparrow}\,\land\mathsf{I}_\uparrow}{A \land A \uparrow ?\ \downarrow ?}\,\lor\mathsf{E}^{x,y}_\downarrow}{A\!\downarrow\uparrow}\,\land\mathsf{E}_{\mathsf{L}\downarrow}}{(A \lor A) \supset A\!\uparrow}\,\supset\mathsf{I}^z_\uparrow$$

Here and henceforth, we abbreviate the use of the rule $\dfrac{A\!\downarrow}{A\!\uparrow}\downarrow\uparrow$ as $A\!\downarrow\uparrow$.

The example suggests that in order to reduce an arbitrary proof to a normal proof, we need another strategy for transforming proofs involving disjunction $\lor$ and falsehood $\bot$. It turns out that we need a *commuting conversion*:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \lor B\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{A\ \mathit{true}}\,x\\ \vdots\\ C\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}\,y\\ \vdots\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\,\lor\mathsf{E}^{x,y}}{C'\ \mathit{true}}\,R \quad \Longrightarrow_C \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A \lor B\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{A\ \mathit{true}}\,x\\ \vdots\\ \dfrac{C\ \mathit{true}}{C'\ \mathit{true}}\,R\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}\,y\\ \vdots\\ \dfrac{C\ \mathit{true}}{C'\ \mathit{true}}\,R\end{array}}{C'\ \mathit{true}}\,\lor\mathsf{E}^{x,y}$$

Here the rule R is assumed to be an elimination rule, since there is no point in applying a commuting conversion when the rule R is an introduction rule. Note that a commuting conversion allows us to effectively ignore the elimination rule $\vee\mathsf{E}$ lying *between* the rule for proving $C\ \mathit{true}$ in the second or third premise and the rule R for proving $C'\ \mathit{true}$ from $C\ \mathit{true}$ in the conclusion. In a certain sense, the only role that the conclusion in the rule $\vee\mathsf{E}$ plays is to indicate that both hypotheses $\overline{A\ \mathit{true}}^{\,x}$ and $\overline{B\ \mathit{true}}^{\,y}$ lead to the same conclusion $C\ \mathit{true}$, instead of two different conclusions, say $C_1\ \mathit{true}$ and $C_2\ \mathit{true}$. In other words, $C\ \mathit{true}$ in the conclusion makes no contribution to the proof because it is the two premises that actually prove $C\ \mathit{true}$. Therefore the rule $\vee\mathsf{E}$ may be ignored as far as deducing another judgment from $C\ \mathit{true}$ in the conclusion is concerned. If we chose the following elimination rule for $\vee$ with a side condition that both hypotheses $\overline{A\ \mathit{true}}^{\,x}$ and $\overline{B\ \mathit{true}}^{\,y}$ lead to the same conclusion, no commuting conversion would be necessary:

$$\begin{array}{c} \underline{\qquad A \vee B\ \mathit{true} \qquad} \ \vee\mathsf{E}^{x,y} \\ \overline{A\ \mathit{true}}^{\,x} \quad \overline{B\ \mathit{true}}^{\,y} \\ \vdots \qquad\qquad \vdots \\ C\ \mathit{true} \quad C\ \mathit{true} \end{array}$$

Now applying a commuting conversion to the proof of $(A \vee A) \supset A\ \mathit{true}$ shown above yields another proof of the same judgment, to which a local reduction can be applied:

$$\cdots \quad \Longrightarrow_C \quad \dfrac{\dfrac{\overline{A \vee A\ \mathit{true}}^{\,z} \quad \dfrac{\dfrac{\overline{A\ \mathit{true}}^{\,x} \quad \overline{A\ \mathit{true}}^{\,x}}{A \wedge A\ \mathit{true}}\wedge\mathsf{I}}{A\ \mathit{true}}\wedge\mathsf{E_L} \quad \dfrac{\dfrac{\overline{A\ \mathit{true}}^{\,y} \quad \overline{A\ \mathit{true}}^{\,y}}{A \wedge A\ \mathit{true}}\wedge\mathsf{I}}{A\ \mathit{true}}\wedge\mathsf{E_L}}{A\ \mathit{true}}\vee\mathsf{E}^{x,y}}{(A \vee A) \supset A\ \mathit{true}}\supset\mathsf{I}^z$$

After removing the two detours in it, we obtain a normal proof annotated with arrows $\downarrow$ and $\uparrow$:

$$\dfrac{\dfrac{\overline{A \vee A\downarrow}^{\,z} \quad \overline{A\downarrow\uparrow}^{\,x} \quad \overline{A\downarrow\uparrow}^{\,y}}{A\uparrow}\vee\mathsf{E}_{\downarrow}^{x,y}}{(A \vee A) \supset A\uparrow}\supset\mathsf{I}_{\uparrow}^{z}$$

1.8 Long normal proofs

While the normalization theorem guarantees the existence of a proof of $A\uparrow$ for every proof of $A\ \mathit{true}$, it does not address the uniqueness of proofs of $A\uparrow$. In fact, such a proof of $A\uparrow$ is not always unique! To see why, observe that the rule $\dfrac{A\downarrow}{A\uparrow}\downarrow\uparrow$ has no restriction on proposition A. Therefore, if a proof of $A\downarrow$ is given where A is not an atomic proposition (*e.g.*, $A = A_1 \supset A_2$), we may either appeal to the rule $\downarrow\uparrow$ to deduce $A\uparrow$ immediately, or apply an elimination rule to $A\downarrow$ to later build a proof of $A\uparrow$ by applying an introduction rule. For example, we may prove $(A \supset B) \supset (A \supset B)\uparrow$ by applying the rule $\downarrow\uparrow$ to $A \supset B\downarrow$:

$$\dfrac{\dfrac{\overline{A \supset B\downarrow}^{\,x}}{A \supset B\uparrow}\downarrow\uparrow}{(A \supset B) \supset (A \supset B)\uparrow}\supset\mathsf{I}_{\uparrow}^{x}$$

Alternatively we may prove the same judgment $(A \supset B) \supset (A \supset B)\uparrow$ by decomposing $A \supset B\downarrow$ until the rule $\downarrow\uparrow$ is applied to $B\downarrow$ for an atomic proposition B:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \supset B\downarrow}^{\,x} \qquad \dfrac{\overline{A\downarrow}^{\,y}}{A\uparrow}\,\downarrow\uparrow}{B\downarrow}\,\supset\!\mathsf{E}_{\downarrow}}{B\uparrow}\,\downarrow\uparrow}{A \supset B\uparrow}\,\supset\!\mathsf{I}^{y}_{\uparrow}}{(A \supset B) \supset (A \supset B)\uparrow}\,\supset\!\mathsf{I}^{x}_{\uparrow}$$

If we require that proposition A in the rule $\downarrow\uparrow$ be atomic, top-down applications of elimination rules meet bottom-up applications of introduction rules only through atomic propositions. Thus every normal proof applies elimination rules until only neutral judgments $A\downarrow$ for atomic propositions A remain, and starts to apply introduction rules only to normal judgments $A\uparrow$ for atomic propositions A. We call such normal proofs as *long normal proofs*. For example, the second proof of $(A \supset B) \supset (A \supset B)\uparrow$ shown above is a long normal proof while the first proof is not.

Now consider the system in Figure 1.3 in which proposition A in the rule $\downarrow\uparrow$ is required to be atomic:

$$\dfrac{A\downarrow}{A\uparrow}\,\downarrow\uparrow\ \textit{(A atomic)}$$

If we can show that the original rule $\downarrow\uparrow$ (without the requirement on proposition A) is derivable, all the elimination rules in the system are strong enough in the sense that even if all propositions are decomposed into atomic propositions by elimination rules, no knowledge is essentially lost. (Here it helps to think of $A\uparrow$ and $A\downarrow$ as expressing a particular strategy for proving $A\ \mathit{true}$.) As a property of *all* elimination rules collectively, it is called the *global completeness* property. (Recall that the local completeness property states that a *specific* elimination rule is strong enough.)

The system in Figure 1.3 satisfies the global completeness property. We inductively show that the original rule $\downarrow\uparrow$ without the requirement on proposition A is derivable.

Proposition 1.14. *The rule* $\dfrac{A\downarrow}{A\uparrow}$ *is derivable.*

Proof. By induction on the structure of proposition A. If A is atomic, we apply the new rule $\downarrow\uparrow$ (with the requirement on proposition A). We show the case $A = A_1 \supset A_2$:

$$\dfrac{\dfrac{\dfrac{A_1 \supset A_2\downarrow \qquad \dfrac{\overline{A_1\downarrow}^{\,x}}{A_1\uparrow}\,\textit{IH on } A_1}{A_2\downarrow}\,\supset\!\mathsf{E}_{\downarrow}}{A_2\uparrow}\,\textit{IH on } A_2}{A_1 \supset A_2\uparrow}\,\supset\!\mathsf{I}^{x}_{\uparrow}$$

□