Chapter 1

Sequent Calculus

This chapter presents a *sequent calculus* for propositional logic. Although we set out to develop it as a device for proving the completeness property of normal proofs (Theorem **??**), the sequent calculus also serves as a basis for proof search strategies implemented in theorem provers. Due to its important role in logic, a sequent calculus is not viewed as a secondary system derivable from a corresponding natural deduction system. Rather it is accepted as a valid formulation of a system of logic in itself, whether a corresponding natural deduction system has been formulated or not.

1.1 Sequent calculus for propositional logic

The sequent calculus for propositional logic consists of inference rules for *sequents* of the form $\Gamma \longrightarrow C$ where Γ is an unordered collection of propositions. Conceptually a sequent $A_1, \cdots, A_n \longrightarrow C$ becomes evident by a proof of $C\!\uparrow$ using neutral judgments $A_1\!\downarrow, \cdots, A_n\!\downarrow$:

$$A_1, \cdots, A_n \longrightarrow C \quad \Longleftrightarrow \quad \begin{array}{ccc} A_1\!\downarrow & \cdots & A_n\!\downarrow \\ & \ddots\;\cdot\;\cdot^{\cdot} & \\ & C\!\uparrow & \end{array}$$

Note that the exchange rule is built into sequents because Γ in a sequent $\Gamma \longrightarrow C$ is an unordered collection of propositions.

It is important that unlike a hypothetical judgment $\Gamma \vdash J$ in which a judgment $J \in \Gamma$ is interpreted as a *hypothesis* $\overline{J}$, a proposition $A \in \Gamma$ in a sequent $\Gamma \longrightarrow C$ denotes just a *neutral judgment* $A\!\downarrow$, which may happen to originate from a hypothesis $\overline{A\!\downarrow}$, but not necessarily. For example, both proofs of $C\!\uparrow$ shown below make evident the same sequent $A \wedge B, A \longrightarrow C$:

$$\begin{array}{c} \dfrac{\overline{A \wedge B\!\downarrow}}{A\!\downarrow}\,\wedge\mathsf{E_{L\downarrow}} \\ \vdots \\ C\!\uparrow \end{array} \qquad\qquad \begin{array}{ccc} \overline{A \wedge B\!\downarrow} & & \overline{A\!\downarrow} \\ & \ddots\;\cdot\;\cdot^{\cdot} & \\ & C\!\uparrow & \end{array}$$

In the left proof, $A \wedge B\!\downarrow$, as well as $A\!\downarrow$, is available as a neutral judgment because the same hypothesis may be used more than once. In the right proof, both neutral judgments $A \wedge B\!\downarrow$ and $A\!\downarrow$ happen to originate from hypotheses. Still, however, we may think of $A \in \Gamma$ in a sequent $\Gamma \longrightarrow C$ as denoting a hypothesis $\overline{A\!\downarrow}$ available in the proof of $C\!\uparrow$, since as far as the proof of $C\!\uparrow$ is concerned, using $A\!\downarrow$ as a neutral judgment or as a hypothesis makes no difference.

An advantage of the sequent calculus over the natural deduction system consists in the fact that a proof of $\Gamma \longrightarrow C$ always proceeds in a bottom-up way, which implies that every inference rule in the sequent calculus is best read in a bottom-up way. (This is not the case in the natural deduction system because every elimination rule is best read in a top-down way.) Consider a sequent $A_1, \cdots, A_i, \cdots, A_n \longrightarrow C$:

$$A_1, \cdots, A_i, \cdots, A_n \longrightarrow C \quad \Longleftrightarrow \quad \begin{array}{ccccc} A_1 \downarrow & \cdots & A_i \downarrow & \cdots & A_n \downarrow \\ & \ddots & & \iddots & \\ & & C \uparrow & & \end{array}$$

For the sake of simplicity, let us assume that an introduction rule applied to $C\uparrow$ produces a new goal $C'\uparrow$ without producing a new hypothesis (as is the case for the rule $\vee\mathsf{I}_{\mathsf{L}\uparrow}$), and that an elimination rule applied to $A_i\downarrow$ produces a new neutral judgment $A_i'\downarrow$ without requiring a separate proof of a normal judgment (as is the case for the rule $\wedge\mathsf{E}_{\mathsf{L}\downarrow}$). If we choose to apply an introduction rule to $C\uparrow$, a new goal $C'\uparrow$ is produced. Thus we now have to prove $A_1, \cdots, A_i, \cdots, A_n \longrightarrow C'$:

$$\frac{A_1, \cdots, A_i, \cdots, A_n \longrightarrow C'}{A_1, \cdots, A_i, \cdots, A_n \longrightarrow C} \quad \Longleftrightarrow \quad \begin{array}{ccccc} A_1 \downarrow & \cdots & A_i \downarrow & \cdots & A_n \downarrow \\ & \ddots & & \iddots & \\ & & \dfrac{C' \uparrow}{C \uparrow} & & \end{array}$$

Such an inference rule in the sequent calculus is called a *right rule* because it focuses on the right side C in a sequent $\Gamma \longrightarrow C$. A right rule then corresponds to an introduction rule in the natural deduction system. If we choose to apply an elimination rule to $A_i\downarrow$, a new neutral judgment $A_i'\downarrow$ is produced while the goal $C\uparrow$ remains the same. Thus we now have to prove $A_1, \cdots, A_i, A_i', \cdots, A_n \longrightarrow C$:

$$\frac{A_1, \cdots, A_i, A_i', \cdots, A_n \longrightarrow C}{A_1, \cdots, A_i, \cdots, A_n \longrightarrow C} \quad \Longleftrightarrow \quad \begin{array}{ccccc} A_1 \downarrow & \cdots & \dfrac{A_i \downarrow}{A_i' \downarrow} & \cdots & A_n \downarrow \\ & \ddots & & \iddots & \\ & & C \uparrow & & \end{array}$$

Such an inference rule in the sequent calculus is called a *left rule* because it focuses on a proposition in the left side Γ in a sequent $\Gamma \longrightarrow C$. A left rule then corresponds to an elimination rule in the natural deduction system.

Keeping in mind the intuition behind sequents, let us consider each inference rule in the sequent calculus. The first rule is an axiom which deals with *initial sequents* of the form $\Gamma, A \longrightarrow A$:

$$\frac{}{\Gamma, A \longrightarrow A}\ \mathit{Init}$$

Note that the rule *Init* corresponds not to the rule Hyp but to the rule $\downarrow\!\uparrow$ in the natural deduction system: it is *not* a rule using a hypothesis; rather it is a rule deducing $A\uparrow$ from $A\downarrow$.

For conjunction $\wedge$, we need two left rules $\wedge L_L$ and $\wedge L_R$, corresponding to the elimination rules $\wedge\mathsf{E}_{\mathsf{L}\downarrow}$ and $\wedge\mathsf{E}_{\mathsf{R}\downarrow}$, and one right rule $\wedge R$, corresponding to the introduction rule $\wedge\mathsf{I}_{\uparrow}$:

$$\frac{\Gamma, A \wedge B, A \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L_L \qquad \frac{\Gamma, A \wedge B, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L_R \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R$$

For implication $\supset$, we need one left rule, corresponding to the elimination rule $\supset\mathsf{E}_{\downarrow}$, and one right rule, corresponding to the introduction rule $\supset\mathsf{I}_{\uparrow}$. Suppose that we wish to prove $\Gamma, A \supset B \longrightarrow C$ by focusing on $A \supset B$ in the left side:

$$\Gamma, A \supset B \longrightarrow C \quad \Longleftrightarrow \quad \begin{array}{ccc} \Gamma_{\downarrow} & \cdots & A \supset B \downarrow \\ & \ddots\ \iddots & \\ & C \uparrow & \end{array}$$

Here $\Gamma_\downarrow$ is a shorthand for $\{A\!\downarrow \mid A \in \Gamma\}$. In order to apply the rule $\supset\!\mathsf{E}_\downarrow$ to $A \supset B\!\downarrow$, we first have to build a proof of $A\!\uparrow$ using $\Gamma_\downarrow$ and $A \supset B\!\downarrow$, which means that we need a proof of $\Gamma, A \supset B \longrightarrow A$:

$$\dfrac{\Gamma, A \supset B \longrightarrow A \quad \cdots}{\Gamma, A \supset B \longrightarrow C} \quad\Longleftrightarrow\quad \begin{array}{c} \Gamma_\downarrow \quad \cdots \quad A \supset B\!\downarrow \\ \ddots\;\cdot^{\cdot^{\cdot}} \\ C\!\uparrow \end{array} \qquad \begin{array}{c} \Gamma_\downarrow \quad \cdots \quad A \supset B\!\downarrow \\ \ddots\;\cdot^{\cdot^{\cdot}} \\ A\!\uparrow \end{array}$$

Then a new neutral judgment $B\!\downarrow$ becomes available for the proof of $C\!\uparrow$, which means that it now suffices to prove $\Gamma, A \supset B, B \longrightarrow C$:

$$\dfrac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C} \quad\Longleftrightarrow\quad \begin{array}{c} \Gamma_\downarrow \quad \cdots \quad \dfrac{A \supset B\!\downarrow \qquad \begin{array}{c} \Gamma_\downarrow \quad \cdots \quad A \supset B\!\downarrow \\ \ddots\;\cdot^{\cdot^{\cdot}} \\ A\!\uparrow \end{array}}{B\!\downarrow} \supset\!\mathsf{E}_\downarrow \\ \ddots\;\cdot^{\cdot^{\cdot}} \\ C\!\uparrow \end{array}$$

Thus we obtain the following left rule $\supset\!L$; the right rule $\supset\!R$ is obtained by a similar analysis:

$$\dfrac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C} \supset\!L \qquad \dfrac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset\!R$$

For disjunction $\vee$, we need one left rule $\vee L$, corresponding to the elimination rule $\vee\mathsf{E}_\downarrow$, and two right rules $\vee R_L$ and $\vee R_R$, corresponding to the introduction rules $\vee\mathsf{I}_{\mathsf{L}\uparrow}$ and $\vee\mathsf{I}_{\mathsf{R}\uparrow}$; the rule $\vee L$ is obtained in a similar way to the rule $\supset\!L$:

$$\dfrac{\Gamma, A \vee B, A \longrightarrow C \quad \Gamma, A \vee B, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee L \qquad \dfrac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B} \vee R_L \qquad \dfrac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B} \vee R_R$$

The rule $\vee L$ is designed in such a way that commuting conversion is built into the sequent calculus. To see why, observe that $\Gamma, A \vee B \longrightarrow C$ in the conclusion describes a proof of the goal $C\!\uparrow$ in which the rule $\vee\mathsf{E}$ is to be applied to $A \vee B\!\downarrow$. Then $\Gamma, A \vee B, A \longrightarrow C$ and $\Gamma, A \vee B, B \longrightarrow C$ in the premises indicate that the rule $\vee\mathsf{E}$ applied to $A \vee B\!\downarrow$ uses the same goal $C\!\uparrow$ in its conclusion. According to the rule $\vee L$, therefore, the conclusion in any instance of the rule $\vee\mathsf{E}$ in the natural deduction system is always the current goal, which means that commuting conversion is built into the sequent calculus.

For truth $\top$, we need a right rule $\top R$ corresponding to the introduction rule $\top\mathsf{I}$, but no left rule (because there is no elimination rule for $\top$); for falsehood $\bot$, we need a left rule $\bot L$ corresponding to the elimination rule $\bot\mathsf{E}$, but no right rule (because there is no introduction rule for $\bot$):

$$\dfrac{}{\Gamma \longrightarrow \top} \top R \qquad \dfrac{}{\Gamma, \bot \longrightarrow C} \bot L$$

Figure 1.1 shows all the inference rules in the sequent calculus for propositional logic. Note that each rule R focuses on a proposition in the sequent of the conclusion, which appears in the left side if R is a left rule, in the right side if R is a right rule, and in both sides if R is the rule *Init*. For example, the rule $\wedge L_L$ focuses on $A \wedge B$ in the left side, the rule $\wedge R$ on $A \wedge B$ in the right side, and the rule *Init* on A. We refer to such a proposition as the *principal formula* of the rule.

The rules $\neg L$ and $\neg R$ are obtained from the notational definition $\neg A = A \supset \bot$. In particular, the rule $\neg L$ implicitly uses the rule $\bot L$:

$$\dfrac{\Gamma, A \supset \bot \longrightarrow A \quad \dfrac{}{\Gamma, A \supset \bot, \bot \longrightarrow C} \bot L}{\Gamma, A \supset \bot \longrightarrow C} \supset\!L$$

$$\frac{}{\Gamma, A \longrightarrow A}\ \mathit{Init} \quad \frac{\Gamma, A \wedge B, A \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L_L \quad \frac{\Gamma, A \wedge B, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L_R \quad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R$$

$$\frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L \quad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R$$

$$\frac{\Gamma, A \vee B, A \longrightarrow C \quad \Gamma, A \vee B, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_L \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_R$$

$$\frac{}{\Gamma \longrightarrow \top}\ \top R \quad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L \quad \frac{\Gamma, \neg A \longrightarrow A}{\Gamma, \neg A \longrightarrow C}\ \neg L \quad \frac{\Gamma, A \longrightarrow \bot}{\Gamma \longrightarrow \neg A}\ \neg R$$

Figure 1.1: Sequent calculus for propositional logic

As in the natural deduction system, the weakening and contraction properties allow us to use a proposition $A \in \Gamma$ zero times and more than once, respectively, in a proof of $\Gamma \longrightarrow C$. Note that the structural properties allow us to identify two sequents $\Gamma \longrightarrow C$ and $\Gamma' \longrightarrow C$ if Γ and Γ' are equivalent as sets (rather than as multisets), *i.e.*, $\{A \mid A \in \Gamma\} = \{A \mid A \in \Gamma'\}$.

Proposition 1.1 (Structural properties).
$\Gamma_\downarrow = \{A\!\downarrow \mid A \in \Gamma\}$ *is a collection of neutral judgments.*
(Weakening) *If* $\Gamma \longrightarrow C$*, then* $\Gamma, A \longrightarrow C$*.*
(Contraction) *If* $\Gamma, A, A \longrightarrow C$*, then* $\Gamma, A \longrightarrow C$*.*

Proof. By induction on the structure of the proof of $\Gamma \longrightarrow C$ and $\Gamma, A, A \longrightarrow C$. □

The sequent calculus in Figure 1.1 satisfies the *subformula property* that every proposition (or formula) in the premise of a rule is a subformula of a certain proposition (or formula) in the conclusion, where the subformula relation is defined as follows: (1) A is a subformula of A; (2) A and B are subformulae of $A \supset B$, $A \wedge B$, and $A \vee B$. For example, the premise of the rule $\wedge L_L$ introduces a new proposition A, which is a subformula of $A \wedge B$ in the conclusion; the premises of the rule $\wedge R$ introduce two new propositions A and B, both of which are subformulae of $A \wedge B$ in the conclusion.

Because of the subformula property, a proof of $\Gamma \longrightarrow C$ needs to consider only subformulae of those propositions in Γ or C. For example, a proof of $\cdot \longrightarrow A \supset (B \supset C)$ never involves an analysis of $A \supset B$ by applying the rule $\supset L$ or $\supset R$ because it is not a subformula of $A \supset (B \supset C)$. In conjunction with the structural properties, therefore, the subformula property implies that the sequent calculus in Figure 1.1 is decidable: there exists a procedure for deciding whether $\Gamma \longrightarrow C$ is provable or not. Intuitively a proof of $\Gamma \longrightarrow C$ generates a finite number of sequents because only a finite number of propositions need to be considered.

Proposition 1.2. *The sequent calculus in Figure 1.1 is decidable.*

Proof. Let us write $\Gamma^\star$ for a set $\{A \mid A \in \Gamma\}$ consisting of elements of a multiset Γ. By the structural properties, $\Gamma \longrightarrow C$ is provable if and only if $\Gamma^\star \longrightarrow C$ is provable. When proving a sequent, therefore, we implicitly use only those sequents of the form $\Gamma^\star \longrightarrow C$ in which no proposition appears more than once in $\Gamma^\star$.

Suppose that we wish to check the provability of the goal sequent $\Gamma \longrightarrow C$. First we generate the set S of all possible sequents using subformulae of propositions in Γ and C. S must be a finite set because of the subformula property. (If Γ and C produce n different subformulae, there are a total of $2^n \times n$ sequents in S.) Next we check each sequent in S and mark it as "proven" if it is provable by the rule Init, $\top R$, or $\bot L$, which are the rules with no premise. Then, for each rule except Init, $\top R$, or $\bot L$, we consider all possible combinations of those sequents marked as "proven" for its premise, and mark as "proven" the sequent corresponding to the conclusion if it is not marked as "proven" yet. For example, for the rule $\wedge L_L$, we mark every sequent of the form $\Gamma, A \wedge B \longrightarrow C$ as "proven" if $\Gamma, A \wedge B, A \longrightarrow C$ is already marked as "proven." Similarly, for the rule $\supset L$, we mark every sequent of the form $\Gamma, A \supset B \longrightarrow C$ as "proven" if $\Gamma, A \supset B \longrightarrow A$

and $\Gamma, A \supset B, B \longrightarrow C$ are already marked as "proven." We repeat the procedure until no more sequent in S can be marked as "proven." The procedure must eventually terminate because the number of combinations of the rules and the sequents in S is finite. If the goal sequent is marked as "proven," we decide that it is provable; otherwise it is not provable. (The procedure described above is the basis for a practical proof search technique called the *inverse method*.) □

The soundness and completeness properties of the sequent calculus show that it is equivalent to the natural deduction system for normal judgments.

Theorem 1.3 (Soundness of the sequent calculus). *If* $\Gamma \longrightarrow C$*, then* $\Gamma_{\downarrow} \vdash C\!\uparrow$.

Theorem 1.4 (Completeness of the sequent calculus). *If* $\Gamma_{\downarrow} \vdash C\!\uparrow$*, then* $\Gamma \longrightarrow C$.

Note that while $A\!\downarrow \in \Gamma_{\downarrow}$ in a hypothetical judgment $\Gamma_{\downarrow} \vdash C\!\uparrow$ denotes a hypothesis $\overline{A\!\downarrow}$, $A \in \Gamma$ in a sequent $\Gamma \longrightarrow C$ denotes just a neutral judgment $A\!\downarrow$, which is not necessarily a hypothesis. The discrepancy does not invalidate the two theorems, however, since as far as the proof of $C\!\uparrow$ is concerned, using $A\!\downarrow$ as a hypothesis when it is a neutral judgment, or vice versa, makes no difference.

The proof of the soundness property is straightforward:

Proof of Theorem 1.3. By induction on the structure of the proof of $\Gamma \longrightarrow C$. Below we show three representative cases. We reuse metavariables Γ and C.

Case $\dfrac{}{\Gamma, A \longrightarrow A}\ \mathit{Init}$

$\Gamma_{\downarrow}, A\!\downarrow \vdash A\!\downarrow$ — by the rule $\mathsf{Hyp}_{\downarrow}$

$\Gamma_{\downarrow}, A\!\downarrow \vdash A\!\uparrow$ — by the rule $\downarrow\uparrow$

Case $\dfrac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ {\supset}L$

$\Gamma_{\downarrow}, A \supset B\!\downarrow \vdash A \supset B\!\downarrow$ — by the rule $\mathsf{Hyp}_{\downarrow}$

$\Gamma_{\downarrow}, A \supset B\!\downarrow \vdash A\!\uparrow$ — by induction hypothesis on $\Gamma, A \supset B \longrightarrow A$

$\Gamma_{\downarrow}, A \supset B\!\downarrow \vdash B\!\downarrow$ — by the rule ${\supset}\mathsf{E}_{\downarrow}$

$\Gamma_{\downarrow}, A \supset B\!\downarrow, B\!\downarrow \vdash C\!\uparrow$ — by induction hypothesis on $\Gamma, A \supset B, B \longrightarrow C$

$\Gamma_{\downarrow}, A \supset B\!\downarrow \vdash C\!\uparrow$ — by the substitution principle (Theorem **??**)

Case $\dfrac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ {\supset}R$

$\Gamma_{\downarrow}, A\!\downarrow \vdash B\!\uparrow$ — by induction hypothesis on $\Gamma, A \longrightarrow B$

$\Gamma_{\downarrow} \vdash A \supset B\!\uparrow$ — by the rule ${\supset}\mathsf{I}_{\uparrow}$

□

On the other hand, the proof of the completeness property is not so straightforward. In fact, it is easy to see that a direct proof attempt fails because inference rules for normal judgments use neutral judgments as well, but the theorem does not mention neutral judgments at all. For example, if the proof of $\Gamma_{\downarrow} \vdash C\!\uparrow$ ends with an application of the rule $\downarrow\uparrow$, the premise is $\Gamma_{\downarrow} \vdash C\!\downarrow$, to which induction hypothesis cannot be applied. Therefore we need to generalize the theorem so that a hypothetical judgment $\Gamma_{\downarrow} \vdash A\!\downarrow$ is also related to sequents in a certain way.

Lemma 1.5 below generalizes Theorem 1.4. The proof itself is straightforward, but formulating the statement connecting $\Gamma_{\downarrow} \vdash A\!\downarrow$ to sequents is far from trivial. Theorem 1.4 follows as an immediate consequence of Lemma 1.5.

Lemma 1.5.
If $\Gamma_{\downarrow} \vdash A\!\downarrow$*, then* $\Gamma, A \longrightarrow C$ *implies* $\Gamma \longrightarrow C$.
If $\Gamma_{\downarrow} \vdash C\!\uparrow$*, then* $\Gamma \longrightarrow C$.

Proof. By simultaneous induction on the structure of the proof of $\Gamma_{\downarrow} \vdash A\!\downarrow$ and $\Gamma_{\downarrow} \vdash C\!\uparrow$. Below we reuse metavariables Γ, A, and C.

Case $\dfrac{}{\Gamma_\downarrow, A\!\downarrow \vdash A\!\downarrow}\ \mathsf{Hyp}_\downarrow$

$\Gamma, A, A \longrightarrow C$ — assumption

$\Gamma, A \longrightarrow C$ — by contraction

Case $\dfrac{\Gamma_\downarrow, A\!\downarrow \vdash B\!\uparrow}{\Gamma_\downarrow \vdash A \supset B\!\uparrow}\ {\supset}\mathsf{I}_\uparrow$

$\Gamma, A \longrightarrow B$ — by induction hypothesis on $\Gamma_\downarrow, A\!\downarrow \vdash B\!\uparrow$

$\Gamma \longrightarrow A \supset B$ — by the rule $\supset R$

Case $\dfrac{\Gamma_\downarrow \vdash A \supset B\!\downarrow \quad \Gamma_\downarrow \vdash A\!\uparrow}{\Gamma_\downarrow \vdash B\!\downarrow}\ {\supset}\mathsf{E}_\downarrow$

$\Gamma, B \longrightarrow C$ — assumption

$\Gamma, A \supset B, B \longrightarrow C$ — by weakening

$\Gamma \longrightarrow A$ — by induction hypothesis on $\Gamma_\downarrow \vdash A\!\uparrow$

$\Gamma, A \supset B \longrightarrow A$ — by weakening

$\Gamma, A \supset B \longrightarrow C$ — by the rule $\supset L$ on $\Gamma, A \supset B \longrightarrow A$ and $\Gamma, A \supset B, B \longrightarrow C$

$\Gamma \longrightarrow C$ — by induction hypothesis on $\Gamma_\downarrow \vdash A \supset B\!\downarrow$ with $\Gamma, A \supset B \longrightarrow C$

Case $\dfrac{\Gamma_\downarrow \vdash A\!\downarrow}{\Gamma_\downarrow \vdash A\!\uparrow}\ {\downarrow\!\uparrow}$

$\Gamma, A \longrightarrow A$ — by the rule *Init*

$\Gamma \longrightarrow A$ — by induction hypothesis on $\Gamma_\downarrow \vdash A\!\downarrow$ with $\Gamma, A \longrightarrow A$

□

The proof of Lemma 1.5 is *constructive*, as opposed to *declarative*, in the sense that it gives an algorithm for converting a proof of $\Gamma_\downarrow \vdash C\!\uparrow$ into a proof of $\Gamma \longrightarrow C$. The key to understanding its constructive nature is to observe that when converting $\Gamma_\downarrow \vdash A\!\downarrow$, a proof of $\Gamma, A \longrightarrow C$ for some proposition C is given as an assumption so that a new proof of $\Gamma \longrightarrow C$ is produced. For example, the case $\dfrac{\Gamma_\downarrow \vdash A\!\downarrow}{\Gamma_\downarrow \vdash A\!\uparrow}\ {\downarrow\!\uparrow}$ creates a proof of $\Gamma, A \longrightarrow A$ using the rule *Init*, which then serves as an assumption in converting the premise $\Gamma_\downarrow \vdash A\!\downarrow$. The case $\dfrac{}{\Gamma_\downarrow, A\!\downarrow \vdash A\!\downarrow}\ \mathsf{Hyp}_\downarrow$ uses the contraction property to deduce $\Gamma, A \longrightarrow C$ from such an assumption $\Gamma, A, A \longrightarrow C$.

1.2 Cut elimination

We have seen that in the natural deduction system based on hypothetical judgments, reflexivity and the substitution principle confirm that the system adheres to the definition of hypothetical judgments. That is, failure of reflexivity or the substitution principle indicates the existence of an inference rule that does not respect the definition of hypothetical judgments as concise representations of hypothetical proofs.

In the case of the sequent calculus, we may test its integrity by checking two similar principles, especially in view of the fact that $A \in \Gamma$ in a sequent $\Gamma \longrightarrow C$ can be thought of as denoting a hypothesis $\overline{A\!\downarrow}$. The first, corresponding to reflexivity in the natural deduction system, is the provability of every initial sequent $\Gamma, A \longrightarrow A$, which directly follows from the rule *Init*. The second, corresponding to the substitution principle, is the *admissibility of the cut rule* (where the cut rule is another rule to be explained later):

Theorem 1.6 (Admissiblity of the cut rule). *If $\Gamma \longrightarrow A$ and $\Gamma, A \longrightarrow C$, then $\Gamma \longrightarrow C$.*

Thus the admissibility of the cut rule is to the sequent calculus what the substitution principle is to the natural deduction system: if the substitution principle fails, it indicates that the natural deduction system is not sound (or even non-sense); similarly if the admissibility of the cut rule fails, it indicates that the sequent calculus is not sound (or even non-sense).

Theorem 1.6 implies that if a new rule $\dfrac{A\uparrow}{A\downarrow}\ \upharpoonleft\!\downharpoonright$ is added to the natural deduction system for normal and neutral judgments, we can safely remove any occurrence of the rule $\upharpoonleft\!\downharpoonright$ in a proof of $C\uparrow$. The intuition is that Theorem 1.6 may be rewritten in terms of normal and neutral judgments as follows:

$$\textit{If}\quad \begin{array}{c}\Gamma_{\downarrow}\\ \vdots\\ \dfrac{A\uparrow}{A\downarrow}\ \upharpoonleft\!\downharpoonright\\ \vdots\\ C\uparrow\end{array}\quad \textit{, then}\quad \begin{array}{c}\Gamma_{\downarrow}\\ \vdots\\ C\uparrow\end{array}\ .$$

Since an occurrence of the rule $\upharpoonleft\!\downharpoonright$ in a proof of $C\uparrow$ corresponds to a detour in a proof of $C\ true$, Theorem 1.6 implies in turn that we can transform the *entire* proof of $C\ true$ so as to remove any detour in it. In this sense, Theorem 1.6 states that the natural deduction system for truth judgments is *globally sound*. (Recall that the local soundness property states that a detour specific to a connective can be locally eliminated, without transforming the entire proof.) We will later formalize the global soundness property as the normalization theorem (Theorem 1.8).

The proof of Theorem 1.6 proceeds by nested induction on the structure of: 1) proposition A which is called the *cut formula*; 2) proof of $\Gamma \longrightarrow A$; 3) proof of $\Gamma, A \longrightarrow C$. Here are a few examples of applying induction hypothesis in the proof of Theorem 1.6:

- We wish to prove that $\Gamma \longrightarrow A \supset B$ and $\Gamma, A \supset B \longrightarrow C$ imply $\Gamma \longrightarrow C$. Since A is a subformula of the cut formula $A \supset B$, the induction hypothesis on A proves that $\Gamma' \longrightarrow A$ and $\Gamma', A \longrightarrow C'$ imply $\Gamma' \longrightarrow C'$ for any Γ' and C', and also regardless of the structure of the proof of $\Gamma' \longrightarrow A$ and $\Gamma', A \longrightarrow C'$.

- We wish to prove that $\Gamma \longrightarrow A$ and $\Gamma, A \longrightarrow C$ imply $\Gamma \longrightarrow C$. Suppose that the proof of $\Gamma \longrightarrow A$ has the following structure:

  $$\dfrac{\cdots \quad \overset{\mathcal{D}}{\Gamma, B \longrightarrow A}}{\Gamma \longrightarrow A}\ R$$

  Then we weaken $\Gamma, A \longrightarrow C$ to obtain a proof $\mathcal{E}$ of $\Gamma, B, A \longrightarrow C$. Since $\mathcal{D}$ is strictly smaller than the proof of $\Gamma \longrightarrow A$, the induction hypothesis on proposition A, proof $\mathcal{D}$, and proof $\mathcal{E}$ yields $\Gamma, B \longrightarrow C$, irrespective of the structure (or size) of $\mathcal{E}$.

- We wish to prove that $\Gamma \longrightarrow A$ and $\Gamma, A \longrightarrow C$ imply $\Gamma \longrightarrow C$. Suppose that the proof of $\Gamma, A \longrightarrow C$ has the following structure:

  $$\dfrac{\cdots \quad \overset{\mathcal{E}}{\Gamma, B, A \longrightarrow C'}}{\Gamma, A \longrightarrow C}\ R$$

  Then we weaken $\Gamma \longrightarrow A$ to obtain a proof $\mathcal{D}$ of $\Gamma, B \longrightarrow A$, which has exactly the same structure (or size) as the proof of $\Gamma \longrightarrow A$. Since $\mathcal{E}$ is strictly smaller than the proof of $\Gamma, A \longrightarrow C$, the induction hypothesis on proposition A, proof $\mathcal{D}$, and proof $\mathcal{E}$ yields $\Gamma, B \longrightarrow C'$. (Then we typically apply the same rule R to deduce $\Gamma \longrightarrow C$.)

The proof of Theorem 1.6 considers all possible combinations of the last inference rule $R_{\mathcal{D}}$ in the proof $\mathcal{D}$ of $\Gamma \longrightarrow A$ and the last inference rule $R_{\mathcal{E}}$ in the proof $\mathcal{E}$ of $\Gamma, A \longrightarrow C$. The combinations of the rules $R_{\mathcal{D}}$ and $R_{\mathcal{E}}$ are divided as follows:

1. At least one of $R_{\mathcal{D}}$ and $R_{\mathcal{E}}$ is the rule $Init$.

   (a) $R_{\mathcal{D}}$ is the rule $Init$. In this case, we have $\Gamma = \Gamma', A$.

(b) $R_\mathcal{E}$ is the rule *Init*. In this case, we have either $\Gamma = \Gamma', C$ or $A = C$.

2. Neither of $R_\mathcal{D}$ and $R_\mathcal{E}$ is the rule *Init*.

   (a) A is the principal formula of both $R_\mathcal{D}$ and $R_\mathcal{E}$. In this case, $R_\mathcal{D}$ is a right rule and $R_\mathcal{E}$ is a left rule.

   (b) A is not the principal formula of $R_\mathcal{D}$. In this case, $R_\mathcal{D}$ is a left rule.

   (c) A is not the principal formula of $R_\mathcal{E}$. In this case, $R_\mathcal{E}$ can be both a left rule and a right rule.

Note that 1-(a) and 1-(b) overlap because both $R_\mathcal{D}$ and $R_\mathcal{E}$ can be the rule *Init*, and that 2-(b) and 2-(c) overlap because A may be the principal formula of neither $R_\mathcal{D}$ nor $R_\mathcal{E}$.

The proof of Theorem 1.6 is constructive because it gives an algorithm for building a proof of $\Gamma \longrightarrow C$ out of proofs of $\Gamma \longrightarrow A$ and $\Gamma, A \longrightarrow C$. The algorithm is non-deterministic because of the overlapping cases 1-(a) and 1-(b), and 2-(b) and 2-(c).

Proof of Theorem 1.6. By nested induction on the structure of: 1) cut formula A; 2) proof of $\Gamma \longrightarrow A$; 3) proof of $\Gamma, A \longrightarrow C$. Here we consider the fragment of the sequent calculus with the rules *Init*, $\supset L$, and $\supset R$ only.

We write $\mathcal{D} :: J$ or $\overset{\mathcal{D}}{J}$ to say that $\mathcal{D}$ is a proof of J. Let $R_\mathcal{D}$ be the last inference rule in the proof $\mathcal{D}$ of $\Gamma \longrightarrow A$ and $R_\mathcal{E}$ the last inference rule in the proof $\mathcal{E}$ of $\Gamma, A \longrightarrow C$.

Case 1-(a): $R_\mathcal{D}$ is the rule *Init*. We have $\Gamma = \Gamma', A$.

$$\mathcal{D} = \frac{}{\Gamma', A \longrightarrow A}\ Init$$

$\Gamma', A, A \longrightarrow C$	assumption
$\Gamma', A \longrightarrow C$	by contraction
$\Gamma \longrightarrow C$	from $\Gamma = \Gamma', A$

Case 1-(b): $R_\mathcal{E}$ is the rule *Init*.

Subcase: $\Gamma = \Gamma', C$

$$\mathcal{E} = \frac{}{\Gamma', C, A \longrightarrow C}\ Init$$

$\Gamma', C \longrightarrow C$	by the rule *Init*
$\Gamma \longrightarrow C$	from $\Gamma = \Gamma', C$

Subcase: $A = C$

$\Gamma \longrightarrow A$	assumption
$\Gamma \longrightarrow C$	from $A = C$

Case 2-(a): A is the principal formula of both $R_\mathcal{D}$ and $R_\mathcal{E}$. We have $A = A_1 \supset A_2$.

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Gamma, A_1 \longrightarrow A_2}}{\Gamma \longrightarrow A_1 \supset A_2} \supset R \qquad \mathcal{E} = \frac{\overset{\mathcal{E}_1}{\Gamma, A_1 \supset A_2 \longrightarrow A_1} \quad \overset{\mathcal{E}_2}{\Gamma, A_1 \supset A_2, A_2 \longrightarrow C}}{\Gamma, A_1 \supset A_2 \longrightarrow C} \supset L$$

$\mathcal{E}_1' :: \Gamma \longrightarrow A_1$	by IH on $A_1 \supset A_2$, $\mathcal{D}$, and $\mathcal{E}_1$
$\mathcal{D}' :: \Gamma, A_2 \longrightarrow A_1 \supset A_2$	by weakening $\mathcal{D} :: \Gamma \longrightarrow A_1 \supset A_2$
$\mathcal{E}_2' :: \Gamma, A_2 \longrightarrow C$	by IH on $A_1 \supset A_2$, $\mathcal{D}'$, and $\mathcal{E}_2$
$\mathcal{D}_1' :: \Gamma \longrightarrow A_2$	by IH on A_1, $\mathcal{E}_1'$, $\mathcal{D}_1$
$\Gamma \longrightarrow C$	by IH on A_2, $\mathcal{D}_1'$, and $\mathcal{E}_2'$

Case 2-(b): A is not the principal formula of $R_\mathcal{D}$. We have $\Gamma = \Gamma', B_1 \supset B_2$.

$$\mathcal{D} = \frac{\overset{\mathcal{D}_1}{\Gamma', B_1 \supset B_2 \longrightarrow B_1} \quad \overset{\mathcal{D}_2}{\Gamma', B_1 \supset B_2, B_2 \longrightarrow A}}{\Gamma', B_1 \supset B_2 \longrightarrow A} \supset L$$

$\mathcal{E}' :: \Gamma', B_1 \supset B_2, B_2, A \longrightarrow C$ by weakening $\mathcal{E} :: \Gamma', B_1 \supset B_2, A \longrightarrow C$
$\mathcal{E}'' :: \Gamma', B_1 \supset B_2, B_2 \longrightarrow C$ by IH on A, $\mathcal{D}_2$, and $\mathcal{E}'$
$\Gamma', B_1 \supset B_2 \longrightarrow C$ by the rule $\supset L$ with $\mathcal{D}_1$ and $\mathcal{E}''$
$\Gamma \longrightarrow C$ from $\Gamma = \Gamma', B_1 \supset B_2$

Case 2-(c): A is not the principal formula of $R_{\mathcal{E}}$.

Subcase: $\Gamma = \Gamma', B_1 \supset B_2$ where $B_1 \supset B_2$ is the principal formula of $R_{\mathcal{E}}$

$$\mathcal{E} :: \frac{\overset{\mathcal{E}_1}{\Gamma', B_1 \supset B_2, A \longrightarrow B_1} \quad \overset{\mathcal{E}_2}{\Gamma', B_1 \supset B_2, A, B_2 \longrightarrow C}}{\Gamma', B_1 \supset B_2, A \longrightarrow C} \supset L$$

$\mathcal{E}_1' :: \Gamma', B_1 \supset B_2 \longrightarrow B_1$ by IH on A, $\mathcal{D}$, and $\mathcal{E}_1$
$\mathcal{D}' :: \Gamma', B_1 \supset B_2, B_2 \longrightarrow A$ by weakening $\mathcal{D} :: \Gamma \longrightarrow A$ (with $\Gamma = \Gamma', B_1 \supset B_2$)
$\mathcal{E}_2' :: \Gamma', B_1 \supset B_2, B_2 \longrightarrow C$ by IH on A, $\mathcal{D}'$, and $\mathcal{E}_2$
$\Gamma', B_1 \supset B_2 \longrightarrow C$ by the rule $\supset L$ with $\mathcal{E}_1'$ and $\mathcal{E}_2'$
$\Gamma \longrightarrow C$ from $\Gamma = \Gamma', B_1 \supset B_2$

Subcase: $C = C_1 \supset C_2$ is the principal formula of $R_{\mathcal{E}}$

$$\mathcal{E} :: \frac{\overset{\mathcal{E}_1}{\Gamma, A, C_1 \longrightarrow C_2}}{\Gamma, A \longrightarrow C_1 \supset C_2} \supset R$$

$\mathcal{D}' :: \Gamma, C_1 \longrightarrow A$ by weakening $\mathcal{D} :: \Gamma \longrightarrow A$
$\mathcal{E}_1' :: \Gamma, C_1 \longrightarrow C_2$ by IH on A, $\mathcal{D}'$, and $\mathcal{E}_1$
$\Gamma \longrightarrow C_1 \supset C_2$ by the rule $\supset R$ with $\mathcal{E}_1'$
$\Gamma \longrightarrow C$ from $C = C_1 \supset C_2$

□

The admissibility of the cut rule has as a corollary one of the central theorems in the study of logic: *cut elimination* (also called *Hauptsatz* meaning "main theorem"). Consider an extension of the sequent calculus with the cut rule shown below, where we use sequents of the form $\Gamma \overset{+}{\longrightarrow} C$ to distinguish the extended system from the system in Figure 1.1:

$$\frac{\Gamma \overset{+}{\longrightarrow} A \quad \Gamma, A \overset{+}{\longrightarrow} C}{\Gamma \overset{+}{\longrightarrow} C} \; Cut$$

Cut elimination states that the rule Cut is redundant:

Theorem 1.7 (Cut elimination). $\Gamma \longrightarrow C$ *if and only if* $\Gamma \overset{+}{\longrightarrow} C$.

Proof. The *only if* part is trivial. For the *if* part, we proceed by induction on the structure of the proof of $\Gamma \overset{+}{\longrightarrow} C$. The only interesting case is the rule Cut:

Case $\dfrac{\Gamma \overset{+}{\longrightarrow} A \quad \Gamma, A \overset{+}{\longrightarrow} C}{\Gamma \overset{+}{\longrightarrow} C} \; Cut$

$\Gamma \longrightarrow A$ by induction hypothesis on $\Gamma \overset{+}{\longrightarrow} A$
$\Gamma, A \longrightarrow C$ by induction hypothesis on $\Gamma, A \overset{+}{\longrightarrow} C$
$\Gamma \longrightarrow C$ by Theorem 1.6

□

Note that the rule Cut destroys the subformula property: it does not analyze a proposition in the conclusion, so A can be an arbitrary proposition completely unrelated to Γ and C. Thus the presence of the rule Cut makes it difficult to prove a sequent because each application of the rule Cut must "guess" such a proposition A. Fortunately the cut elimination theorem says that the rule Cut can be discarded without sacrificing the expressive power of the sequent calculus.

1.3 Normalization for the natural deduction system

Theorem 1.8 states that for every proof of $A\ \mathit{true}$, there exists a proof of $A\!\uparrow$. We now appeal to the cut elimination theorem to prove the same result, but covering all connectives (including $\vee$ and $\bot$). Our goal is to prove the normalization theorem stated in terms of hypothetical judgments:

Theorem 1.8 (Normalization). *$\Gamma \vdash A\ \mathit{true}$ if and only if $\Gamma_{\downarrow} \vdash A\!\uparrow$.*

To this end, we introduce two *annotated judgments* $\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$ and $\Gamma_{\downarrow} \vdash^{+} A\!\uparrow$, for which we use the following rule in addition to those rules for $\Gamma_{\downarrow} \vdash A\!\downarrow$ and $\Gamma_{\downarrow} \vdash A\!\uparrow$:

$$\frac{\Gamma_{\downarrow} \vdash^{+} A\!\uparrow}{\Gamma_{\downarrow} \vdash^{+} A\!\downarrow}\ \uparrow\!\downarrow$$

As it is based on hypothetical judgments, the new system satisfies the substitution principle (which extends Theorem **??**); we assume that the exchange rule is built-in:

Theorem 1.9 (Substitution).
If $\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$ and $\Gamma_{\downarrow}, A\!\downarrow \vdash^{+} C\!\downarrow$, then $\Gamma_{\downarrow} \vdash^{+} C\!\downarrow$.
If $\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$ and $\Gamma_{\downarrow}, A\!\downarrow \vdash^{+} C\!\uparrow$, then $\Gamma_{\downarrow} \vdash^{+} C\!\uparrow$.

In conjunction with the rule $\downarrow\!\uparrow$, the rule $\uparrow\!\downarrow$ effectively collapses the distinction between $A\!\uparrow$ and $A\!\downarrow$: a proof of $A\!\uparrow$ leads to a proof of $A\!\downarrow$ and vice versa. Thus both $A\!\uparrow$ and $A\!\downarrow$ in the new system are essentially no different from $A\ \mathit{true}$, as stated in the two theorems below; $\Gamma_{\downarrow} = \{A\!\downarrow \mid A\ \mathit{true} \in \Gamma\}$ is a collection of neutral judgments derived from truth judgments in Γ:

Theorem 1.10 (Soundness of the annotated judgments).
If $\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$, then $\Gamma \vdash A\ \mathit{true}$.
If $\Gamma_{\downarrow} \vdash^{+} A\!\uparrow$, then $\Gamma \vdash A\ \mathit{true}$.

Proof. By simultaneous induction on the structure of the proof of $\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$ and $\Gamma_{\downarrow} \vdash^{+} A\!\uparrow$. □

Theorem 1.11 (Completeness of the annotated judgments).
If $\Gamma \vdash A\ \mathit{true}$, then $\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$.
If $\Gamma \vdash A\ \mathit{true}$, then $\Gamma_{\downarrow} \vdash^{+} A\!\uparrow$.

Proof. By induction on the structure of the proof of $\Gamma \vdash A\ \mathit{true}$. We use the rules $\downarrow\!\uparrow$ and $\uparrow\!\downarrow$ to convert between $A\!\downarrow$ and $A\!\uparrow$ whenever necessary. We show two cases.

Case $\dfrac{\Gamma, A_1\ \mathit{true} \vdash A_2\ \mathit{true}}{\Gamma \vdash A_1 \supset A_2\ \mathit{true}}\ {\supset}\mathsf{I}$ where $A = A_1 \supset A_2$

$\Gamma_{\downarrow}, A_1\!\downarrow \vdash^{+} A_2\!\uparrow$ by induction hypothesis on $\Gamma, A_1\ \mathit{true} \vdash A_2\ \mathit{true}$
$\Gamma_{\downarrow} \vdash^{+} A_1 \supset A_2\!\uparrow$ by the rule ${\supset}\mathsf{I}_{\uparrow}$, proving the second clause
$\Gamma_{\downarrow} \vdash^{+} A_1 \supset A_2\!\downarrow$ by the rule $\uparrow\!\downarrow$, proving the first clause

Case $\dfrac{\Gamma \vdash B \supset A\ \mathit{true} \quad \Gamma \vdash B\ \mathit{true}}{\Gamma \vdash A\ \mathit{true}}\ {\supset}\mathsf{E}$

$\Gamma_{\downarrow} \vdash^{+} B \supset A\!\downarrow$ by induction hypothesis on $\Gamma \vdash B \supset A\ \mathit{true}$
$\Gamma_{\downarrow} \vdash^{+} B\!\uparrow$ by induction hypothesis on $\Gamma \vdash B\ \mathit{true}$
$\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$ by the rule ${\supset}\mathsf{E}_{\downarrow}$, proving the first clause
$\Gamma_{\downarrow} \vdash^{+} A\!\uparrow$ by the rule $\downarrow\!\uparrow$, proving the second clause

□

Now we can complete the proof of Theorem 1.8 by showing that $\Gamma \overset{+}{\longrightarrow} A$ and $\Gamma_{\downarrow} \vdash^{+} A\!\uparrow$ are equivalent (Theorems 1.12 and 1.13); we use an appropriate definition of $\Gamma_{\downarrow}$ depending on the definition of Γ:

$$\begin{array}{rcll} \Gamma_{\downarrow} \vdash A\!\uparrow & \Longleftrightarrow & \Gamma \longrightarrow A & \textit{by Theorems 1.3 and 1.4} \\ & \Longleftrightarrow & \Gamma \overset{+}{\longrightarrow} A & \textit{by Theorem 1.7} \\ & \Longleftrightarrow & \Gamma_{\downarrow} \vdash^{+} A\!\uparrow & \textit{by Theorems 1.12 and 1.13} \\ & \Longleftrightarrow & \Gamma \vdash A \; true & \textit{by Theorems 1.10 and 1.11} \end{array}$$

$\Gamma \vdash A\ true$ $\quad\quad$ $\Gamma_{\downarrow} \vdash A\!\uparrow$ / $\Gamma_{\downarrow} \vdash A\!\downarrow$ $\quad\quad$ $\Gamma \longrightarrow A$ $\quad\quad$ $\Gamma \longrightarrow A$

$$\frac{\Gamma_{\downarrow} \vdash^{+} A\!\uparrow}{\Gamma_{\downarrow} \vdash^{+} A\!\downarrow} \;\Uparrow\!\!\Downarrow \qquad\qquad \frac{\Gamma \overset{+}{\longrightarrow} A \quad \Gamma, A \overset{+}{\longrightarrow} C}{\Gamma \overset{+}{\longrightarrow} C} \; Cut$$

The proof of Theorems 1.12 and 1.13 is almost the same as the proof of Theorems 1.3 and 1.4, except for the additional case in which the rule $\Uparrow\!\!\Downarrow$ or Cut is involved. The proof of of Theorem 1.13 follows from a lemma similar to Lemma 1.5.

Theorem 1.12 (Soundness of the sequent calculus with the cut rule). *If* $\Gamma \overset{+}{\longrightarrow} C$, *then* $\Gamma_{\downarrow} \vdash^{+} C\!\uparrow$.

Proof. By induction on the structure of the proof of $\Gamma \overset{+}{\longrightarrow} C$. We show the case for the rule Cut.

Case $\dfrac{\Gamma \overset{+}{\longrightarrow} A \quad \Gamma, A \overset{+}{\longrightarrow} C}{\Gamma \overset{+}{\longrightarrow} C} \; Cut$

$\Gamma_{\downarrow} \vdash^{+} A\!\uparrow$ — by induction hypothesis on $\Gamma \overset{+}{\longrightarrow} A$
$\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$ — by the rule $\Uparrow\!\!\Downarrow$
$\Gamma_{\downarrow}, A\!\downarrow \vdash^{+} C\!\uparrow$ — by induction hypothesis on $\Gamma, A \overset{+}{\longrightarrow} C$
$\Gamma_{\downarrow} \vdash^{+} C\!\uparrow$ — by Theorem 1.9 with $\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$ and $\Gamma_{\downarrow}, A\!\downarrow \vdash^{+} C\!\uparrow$

□

Theorem 1.13 (Completeness of the sequent calculus with the cut rule). *If* $\Gamma_{\downarrow} \vdash^{+} C\!\uparrow$, *then* $\Gamma \overset{+}{\longrightarrow} C$.

Lemma 1.14.
If $\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$, *then* $\Gamma, A \overset{+}{\longrightarrow} C$ *implies* $\Gamma \overset{+}{\longrightarrow} C$.
If $\Gamma_{\downarrow} \vdash^{+} C\!\uparrow$, *then* $\Gamma \overset{+}{\longrightarrow} C$.

Proof. By simultaneous induction on the structure of the proof of $\Gamma_{\downarrow} \vdash^{+} A\!\downarrow$ and $\Gamma_{\downarrow} \vdash^{+} C\!\uparrow$. We show the case for the rule $\Uparrow\!\!\Downarrow$.

Case $\dfrac{\Gamma_{\downarrow} \vdash^{+} A\!\uparrow}{\Gamma_{\downarrow} \vdash^{+} A\!\downarrow} \;\Uparrow\!\!\Downarrow$

$\Gamma \overset{+}{\longrightarrow} A$ — by induction hypothesis on $\Gamma_{\downarrow} \vdash^{+} A\!\uparrow$
$\Gamma, A \overset{+}{\longrightarrow} C$ — assumption
$\Gamma \overset{+}{\longrightarrow} C$ — by the rule Cut with $\Gamma \overset{+}{\longrightarrow} A$ and $\Gamma, A \overset{+}{\longrightarrow} C$

□

Implicit in the proof of Theorem 1.8 is that the proof is constructive: it gives an algorithm for converting a proof of $\Gamma \vdash A\ true$ into a proof of $\Gamma_{\downarrow} \vdash A\!\uparrow$. (Converting a proof of $\Gamma_{\downarrow} \vdash A\!\uparrow$ into a proof of $\Gamma \vdash A\ true$ is trivial.) It is a consequence of constructive proofs of all the theorems involved, in particular Theorem 1.13 (completeness of the sequent calculus with the cut rule) which is generalized to Lemma 1.14, and Theorem 1.6 (admissibility of the cut rule) which is used in the proof of Theorem 1.7 (cut elimination). To be specific, we convert a proof of $\Gamma_{\downarrow} \vdash A\!\uparrow$ into a proof of $\Gamma \vdash A\ true$ as follows:

1. $\Gamma \vdash A\ true$ to $\Gamma_{\downarrow} \vdash^{+} A\!\uparrow$ by Theorem 1.11. We annotate the proof of $\Gamma \vdash A\ true$ by replacing $A\ true$ by $A\!\uparrow$ or $A\!\downarrow$, inserting the rule $\uparrow\!\downarrow$ whenever a detour is encountered.

2. $\Gamma_{\downarrow} \vdash^{+} A\!\uparrow$ to $\Gamma \overset{+}{\longrightarrow} A$ by Theorem 1.13. We insert the rule Cut whenever the rule $\uparrow\!\downarrow$ is encountered.

3. $\Gamma \overset{+}{\longrightarrow} A$ to $\Gamma \longrightarrow A$ by Theorem 1.7. We use the proof of Theorem 1.6 to remove the rule Cut.

4. $\Gamma \longrightarrow A$ to $\Gamma_{\downarrow} \vdash A\!\uparrow$ by Theorem 1.3.

Thus in the heart of the proof of the normalization theorem lies the cut elimination theorem!

A corollary of the normalization theorem (or its proof) is consistency of propositional logic (or first-order logic if universal and existential quantifiers are added): $\bot\ true$ is not provable in propositional logic.

Corollary 1.15 (Consistency). *There is no proof of* $\cdot \vdash \bot\ true$.

Proof. It suffices to show that there is no proof of $\cdot \vdash \bot\!\uparrow$ (by the normalization theorem), or $\cdot \longrightarrow \bot$ (by Theorems 1.3 and 1.4). Since no rule is applicable to $\cdot \longrightarrow \bot$, there is no proof of $\cdot \longrightarrow \bot$. □

Another corollary is that $A \vee B\ true$ is provable only if either $A\ true$ or $B\ true$ is provable.

Corollary 1.16. *If* $\cdot \vdash A \vee B\ true$, *then either* $\cdot \vdash A\ true$ *or* $\cdot \vdash B\ true$.

Proof. $\cdot \vdash A \vee B\ true$ implies $\cdot \longrightarrow A \vee B$, as shown in the proof of the normalization theorem. Since the only way to prove $\cdot \longrightarrow A \vee B$ is by applying either $\vee R_L$ or $\vee R_R$, either $\cdot \longrightarrow A$ or $\cdot \longrightarrow B$ must hold. Therefore either $\cdot \vdash A\ true$ or $\cdot \vdash B\ true$ holds. □

Note, however, that $\Gamma \vdash A \vee B\ true$ does not necessarily imply either $\Gamma \vdash A\ true$ or $\Gamma \vdash B\ true$ if Γ is not empty. For example, $B \vee A\ true \vdash A \vee B\ true$ is provable, but neither $B \vee A\ true \vdash A\ true$ nor $B \vee A\ true \vdash B\ true$ is provable.

Finally constructive logic is shown to be different from classical logic: $A \vee \neg A\ true$, which is called *the law of excluded middle* and is an axiom in classical logic, is not provable in constructive logic.

Corollary 1.17. *There is no proof of* $\cdot \vdash A \vee \neg A\ true$ *for an arbitrary proposition* A.

Proof. If $\cdot \vdash A \vee \neg A\ true$ holds, then either $\cdot \longrightarrow A$ or $\cdot \longrightarrow \neg A$ holds, as shown in the proof of Corollary 1.16. The first sequent is not provable for an arbitrary proposition A. The second sequent is not provable because $A \longrightarrow \bot$ is not provable. □

Note that the law of excluded middle assumes an *arbitrary* proposition A; the use of a specific proposition A makes $A \vee \neg A\ true$ provable. For example, by letting $A = \top$, we obtain $\top \vee \neg \top\ true$, which is certainly provable.

1.4 Contraction and weakening

1.5 Proof terms for the sequent calculus