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CSE-490 Logic in Computer Science 2006
Midterm

	Problem 1	Problem 2	Problem 3	Total
Score				
Max	50	35	15	100

1 Short questions [50 pts]

Answer each question below. All the questions assume constructive logic, not classical logic. For questions 1, 2, and 5, answer either true or false.

1.1 Propositional logic

Question 1. [5 pts]
$A \supset (B \vee C) \equiv (A \supset B) \vee (A \supset C)$ holds. True or false?

Question 2. [5 pts]
$(A \vee B) \supset C \equiv (A \supset C) \wedge (B \supset C)$ holds. True or false?

Question 3. [5 pts]
Give an example of a proof of A *true* that is not normal, but does not contain a detour, either. Use the natural deduction system, but do **not** use hypothetical judgments. You may choose any proposition A you like.

1.2 Proof terms

Question 4. [5 pts]

What is the type of the following proof term?

$$\lambda x : A.\, \mathsf{case}\ \mathsf{inl}_{\bot}\ x\ \mathsf{of}\ \mathsf{inl}\ y \Rightarrow (y, y) \mid \mathsf{inr}\ z \Rightarrow \mathsf{abort}_{A \wedge A}\ y$$

Question 5. [5 pts]

The following proof term represents a **long** normal proof where A and B are atomic propositions. True or false?

$$\lambda x : A \vee B.\, \mathsf{case}\ x\ \mathsf{of}\ \mathsf{inl}\ y \Rightarrow \mathsf{inr}_B\ y \mid \mathsf{inr}\ z \Rightarrow \mathsf{inl}_A\ z$$

Question 6. [5 pts] Find two different proof terms that represent **long** normal proofs of the same judgment $A \supset (A \supset A)\ \mathit{true}$.

Question 7. [5 pts]

Apply β-reductions to reduce the following proof term to the simplest form:

$$\begin{array}{l}(\lambda x\!:\!(A \wedge B) \vee C.\, \lambda y\!:\!(A \wedge B) \supset A.\, \lambda z\!:\!C \supset A. \\ \quad \mathsf{case}\ x\ \mathsf{of}\ \mathsf{inl}\ x_1 \Rightarrow y\ x_1 \mid \mathsf{inr}\ x_2 \Rightarrow z\ x_2) \\ \qquad (\mathsf{inl}_C\ (M, N))\ (\lambda w\!:\!A \wedge B.\, \mathsf{fst}\ w)\end{array}$$

Question 8. [5 pts]

Apply β-reductions and **commuting conversions** to reduce to the following proof term to the simplest form:

$$\begin{array}{l}\lambda w\!:\!A \vee A. \\ \quad \mathsf{fst}\ \ (\mathsf{case}\ (\mathsf{case}\ w\ \mathsf{of}\ \mathsf{inl}\ x \Rightarrow \mathsf{inl}_A\ x \mid \mathsf{inr}\ y \Rightarrow \mathsf{inr}_A\ y)\ \mathsf{of}\ \mathsf{inl}\ x' \Rightarrow (x', x') \mid \mathsf{inr}\ y' \Rightarrow (y', y'))\end{array}$$

1.3 Sequent calculus

Question 9. [5 pts]

Give a proof of a sequent $\cdot \longrightarrow (\neg A \vee B) \supset (A \supset B)$. If not provable, state so.

Question 10. [5 pts]

Give a proof of a sequent $\cdot \longrightarrow (A \supset B) \supset (\neg A \vee B)$. If not provable, state so.

2 Logical equivalence [35 pts]

In the course notes, we use a notational definition of logical equivalence $\equiv$ given as follows:

$$A \equiv B \quad = \quad (A \supset B) \wedge (B \supset A) \; \textit{true}$$

In this problem, we will define $\equiv$ as a logical connective, like $\supset$, $\wedge$, and $\vee$, so that $A \equiv B$ *true* holds if and only if $(A \supset B) \wedge (B \supset A)$ *true* holds. We extend the natural deduction system for propositional logic to incorporate $\equiv$ as a new logical connective orthogonal to the existing logical connectives, by providing its introduction and elimination rules. Thus we assume the following formation rule for $\equiv$:

$$\frac{A \; \textit{prop} \quad B \; \textit{prop}}{A \equiv B \; \textit{prop}} \equiv\mathsf{F}$$

Question 1. [5 pts]
Propose an introduction rule $\equiv\mathsf{I}$ and two elimination rules $\equiv\mathsf{E}_1$ and $\equiv\mathsf{E}_2$ for $\equiv$. Do **<u>not</u>** use hypothetical judgments. Be careful not to destroy the orthogonality of the system.

Question 2. [5 pts]

Show a local reduction $\Longrightarrow_R$ for $\equiv$. (Local soundness)

Question 3. [5 pts]

Show a local expansion $\Longrightarrow_E$ for $\equiv$. (Local completeness)

Question 4. [5 pts]

Rewrite the rules $\equiv \mathsf{I}$, $\equiv \mathsf{E}_1$, $\equiv \mathsf{E}_2$ for neutral and normal judgments by replacing $A\ \mathit{true}$ by $A\!\uparrow$ or $A\!\downarrow$. Call the resultant rules $\equiv \mathsf{I} \uparrow$, $\equiv \mathsf{E} \downarrow_1$, $\equiv \mathsf{E} \downarrow_2$.

Question 5. [5 pts]

Transform the rules $\equiv \mathsf{I} \uparrow$, $\equiv \mathsf{E} \downarrow_1$, $\equiv \mathsf{E} \downarrow_2$ to rules for sequent calculus. Call the resultant rules $\equiv \mathsf{R}$, $\equiv \mathsf{L}_1$, $\equiv \mathsf{L}_2$, respectively. For a collection of propositions in the left side of a sequent, you may use a metavariable Γ.

Question 6. [5 pts]

Assume proofs $A \overset{\mathcal{D}}{\longrightarrow} A$ and $B \overset{\mathcal{E}}{\longrightarrow} B$ to prove $A \equiv B \longrightarrow A \equiv B$. You may refer to proofs of sequents obtained by weakening $A \longrightarrow A$ and $B \longrightarrow B$ as $\mathcal{D}$ and $\mathcal{E}$, respectively. (Global completeness)

Question 7. [5 pts]

Extend the proof of the admissibility of the cut rule with the case for $\equiv$. (Global soundness)

Theorem (Admissiblity of the cut rule). *If* $\Gamma \longrightarrow A$ *and* $\Gamma, A \longrightarrow C$, *then* $\Gamma \longrightarrow C$.

Proof. By nested induction on the structure of: 1) cut formula A; 2) proof of $\Gamma \longrightarrow A$; 3) proof of $\Gamma, A \longrightarrow C$.

Case:

1) $A = A_1 \equiv A_2$.
2) the last inference rule in the proof $\mathcal{D}$ of $\Gamma \longrightarrow A$ is $\equiv\mathsf{R}$.
3) the last inference rule in the proof $\mathcal{E}$ of $\Gamma, A \longrightarrow C$ is $\equiv\mathsf{L}_1$.
4) A is the principal formula of both $\mathcal{D}$ and $\mathcal{E}$.

First show the structure of proofs $\mathcal{D}$ and $\mathcal{E}$.

$\mathcal{D} =$

$\mathcal{E} =$

Then deduce $\Gamma \longrightarrow C$. In each line, show the conclusion in the left side and its justification in the right side.

3 Classical logic [15 pts]

We learned in class that constructive logic "degenerates" to classical logic if the axiom

$$\frac{}{\neg\neg A \supset A \; true} \; \mathsf{DN}$$

is added, where DN stands for 'Double Negation.' Another way to obtain classical logic is by adding either

$$\frac{}{A \vee \neg A \; true} \; \mathsf{EM}$$

or

$$\frac{}{((A \supset B) \supset A) \supset A \; true} \; \mathsf{Pierce}$$

where EM stands for 'Excluded Middle.'

Question 1. [10 pts]

Prove that in the presence of the rule DN, the rule EM is derivable. Use the natural deduction system, but do **not** use hypothetical judgments. Note that A in the rule DN is a metavariable which can be instantiated to any proposition. You may write A instead of $A \; true$ in your proof, if it makes your proof more readable.

Question 2. [5 pts]

Prove that in the presence of the rule EM, the rule Pierce is derivable. Use the natural deduction system, but do **not** use hypothetical judgments. You may write A instead of $A\ \mathit{true}$ in your proof, if it makes your proof more readable.

Question 3. [Extra credit]

The rule DN allows us to conclude $A\ \mathit{true}$ whenever $\neg\neg A\ \mathit{true}$ is provable, or whenever an assumption of $\neg A\ \mathit{true}$ leads to a logical contradiction, as in typical proofs in math textbooks. By the rule EM, we only have to consider two possibilities $A\ \mathit{true}$ and $\neg A\ \mathit{true}$ when proving $C\ \mathit{true}$. Then how can we exploit the rule Pierce when proving $A\ \mathit{true}$?

Work sheet