1 Short questions [50 pts]

Answer each question below. All the questions assume constructive logic, not classical logic. For questions 1, 2, and 5, answer either true or false.

1.1 Propositional logic

Question 1. [5 pts]
\[ A \supset (B \lor C) \equiv (A \supset B) \lor (A \supset C) \] holds. True or false?
False.

Question 2. [5 pts]
\[ (A \lor B) \supset C \equiv (A \supset C) \land (B \supset C) \] holds. True or false?
True.

Question 3. [5 pts]
Give an example of a proof of \( A \text{ true} \) that is not normal, but does not contain a detour, either. Use the natural deduction system, but do not use hypothetical judgments. You may choose any proposition \( A \) you like.
\[
\frac{\frac{\frac{A \lor A}{A \land A} \text{ true} \quad A \text{ true}}{A \land A \text{ true}}}{{(A \lor A) \supset A \text{ true}}} \supset \text{I}^w
\]

1.2 Proof terms

Question 4. [5 pts]
What is the type of the following proof term?
\[
\lambda x : A. \ \text{case} \ x \ \text{of} \ \text{inl} \ y \Rightarrow (y, y) \ \mid \text{inr} \ z \Rightarrow \text{abort}_{A \land A} y
\]
\( A \supset (A \land A) \).

Question 5. [5 pts]
The following proof term represents a long normal proof where \( A \) and \( B \) are atomic propositions. True or false?
\[
\lambda x : A \lor B. \ \text{case} \ x \ \text{of} \ \text{inl} \ y \Rightarrow \text{inr} \ B \ y \ \mid \text{inr} \ z \Rightarrow \text{inl} \ A \ z
\]
True.

Question 6. [5 pts] Find two different proof terms that represent long normal proofs of the same judgment \( A \supset (A \supset A) \text{ true} \).
\[ \lambda x : A. \ \lambda y : A. \ x \ \text{and} \ \lambda x : A. \ \lambda y : A. \ y. \]
Question 7. [5 pts]
Apply $\beta$-reductions to reduce the following proof term to the simplest form:

$$
(\lambda x : (A \wedge B) \lor C. \lambda y : (A \wedge B) \supset A. \lambda z : C \supset A.
\text{case } x \text{ of } \text{inl } x_1 \Rightarrow y x_1 \mid \text{inr } x_2 \Rightarrow z x_2)$$

$$(\text{inl}_C(M, N)) (\lambda w : A \wedge B. \text{fst } w)$$

$$\lambda z : C \supset A. M.$$ 

Question 8. [5 pts]
Apply $\beta$-reductions and commuting conversions to reduce to the following proof term to the simplest form:

$$\lambda w : A \lor A. \text{fst } (\text{case } (\text{case } w \Rightarrow \text{inl } x \mid \text{inr } y) \text{ of } \text{inl } x' \Rightarrow (x', x') \mid \text{inr } y' \Rightarrow (y', y'))$$

$$\lambda w : A \lor A. \text{case } w \Rightarrow x \mid \text{inr } y \Rightarrow y.$$ 

1.3 Sequent calculus

Question 9. [5 pts]
Give a proof of a sequent $\cdot \rightarrow (\lnot A \lor B) \supset (A \supset B)$. If not provable, state so.

$$
\dfrac{
\lnot A \lor B, A, \lnot A \rightarrow A}{\lnot A \lor B, A, \lnot A \rightarrow B} \quad \text{Init}
\dfrac{
\lnot A \lor B, A \rightarrow B}{\lnot A \lor B, A 
\supset A \supset B \quad \supset R
\over \cdot \rightarrow (\lnot A \lor B) \supset (A \supset B) \supset R}

$$

Question 10. [5 pts]
Give a proof of a sequent $\cdot \rightarrow (A \supset B) \supset (\lnot A \lor B)$. If not provable, state so.

Not provable.

2 Logical equivalence [35 pts]

In the course notes, we use a notational definition of logical equivalence $\equiv$ given as follows:

$$A \equiv B = (A \supset B) \land (B \supset A) \text{ true}$$

In this problem, we will define $\equiv$ as a logical connective, like $\supset$, $\land$, and $\lor$, so that $A \equiv B \text{ true}$ holds if and only if $(A \supset B) \land (B \supset A) \text{ true}$ holds. We extend the natural deduction system for propositional logic to incorporate $\equiv$ as a new logical connective orthogonal to the existing logical connectives, by providing its introduction and elimination rules. Thus we assume the following formation rule for $\equiv$:

$$
\dfrac{
A \text{ prop} \quad B \text{ prop}}{A \equiv B \text{ prop} \quad \equiv F}
$$
Question 1. [5 pts]
Propose an introduction rule $\equiv I$ and two elimination rules $\equiv E_1$ and $\equiv E_2$ for $\equiv$. Do not use hypothetical judgments. Be careful not to destroy the orthogonality of the system.

\[
\begin{align*}
\frac{A \text{ true}^x \quad B \text{ true}^y}{A \equiv B \text{ true}} & \equiv I^{x,y} \\
\vdots \quad \vdots \\
\frac{B \text{ true} \quad A \text{ true}}{A \equiv B \text{ true}} & \equiv E_1 \\
\frac{A \equiv B \text{ true} \quad A \text{ true}}{B \text{ true}} & \equiv E_2
\end{align*}
\]

Question 2. [5 pts]
Show a local reduction $\Rightarrow_R$ for $\equiv$. (Local soundness)

\[
\begin{align*}
\frac{A \text{ true}^x \quad B \text{ true}^y}{A \equiv B \text{ true}} & \equiv I^{x,y} \\
\vdots \quad \vdots \\
\frac{B \text{ true} \quad A \text{ true}}{A \equiv B \text{ true}} & \equiv E
\end{align*}
\]

Question 3. [5 pts]
Show a local expansion $\Rightarrow_E$ for $\equiv$. (Local completeness)

\[
\begin{align*}
\frac{A \equiv \mathcal{D} \quad B \equiv \mathcal{D}}{A \equiv B \text{ true}} \Rightarrow_E & \equiv I^{x,y} \\
\frac{A \equiv B \text{ true} \quad A \equiv \mathcal{D}}{B \equiv \mathcal{D}} \equiv E_1 \\
\frac{A \equiv B \text{ true} \quad B \equiv \mathcal{D}}{A \equiv \mathcal{D}} \equiv E_2
\end{align*}
\]

Question 4. [5 pts]
Rewrite the rules $\equiv I$, $\equiv E_1$, $\equiv E_2$ for neutral and normal judgments by replacing $A \text{ true}$ by $A \uparrow$ or $A \downarrow$. Call the resultant rules $\equiv I^{\uparrow}$, $\equiv E_1^{\downarrow}$, $\equiv E_2^{\downarrow}$.

\[
\begin{align*}
\frac{A \uparrow^x \quad B \downarrow^y}{A \equiv B \uparrow \downarrow} & \equiv I^{x,y} \\
\vdots \quad \vdots \\
\frac{B \uparrow \quad A \downarrow}{A \equiv B \uparrow \downarrow} & \equiv E_1 \\
\frac{A \equiv B \downarrow \uparrow \quad A \downarrow}{B \downarrow \uparrow} & \equiv E_2
\end{align*}
\]

Question 5. [5 pts]
Transform the rules $\equiv I^{\uparrow}$, $\equiv E_1^{\downarrow}$, $\equiv E_2^{\downarrow}$ to rules for sequent calculus. Call the resultant rules $\equiv R$, $\equiv L_1$, $\equiv L_2$, respectively. For a collection of propositions in the left side of a sequent, you may use a metavariable $\Gamma$.

\[
\begin{align*}
\frac{\Gamma, A \rightarrow B \quad \Gamma, B \rightarrow A}{\Gamma \rightarrow A \equiv B} & \equiv R \\
\frac{\Gamma, A \equiv B \rightarrow A \quad \Gamma, A \equiv B, B \rightarrow C}{\Gamma, A \equiv B \rightarrow C} \equiv L_1 \\
\frac{\Gamma, A \equiv B \rightarrow C}{\Gamma, A \equiv B \rightarrow C} \equiv L_1
\end{align*}
\]

Question 6. [5 pts]
Assume proofs $\frac{A}{D}$ and $\frac{B}{E}$ to prove $A \equiv B \rightarrow A \equiv B$. You may refer to proofs of sequents obtained by weakening $A \rightarrow A$ and $B \rightarrow B$ as $D$ and $E$, respectively. (Global completeness)

\[
\frac{\frac{A \equiv B, A \rightarrow A}{A \equiv B, A \rightarrow B} \equiv L_1}{A \equiv B \rightarrow A \equiv B}
\frac{\frac{A \equiv B, B \rightarrow B}{A \equiv B, B \rightarrow A} \equiv R}{A \equiv B \rightarrow A \equiv B}
\]

**Question 7. [5 pts]**

Extend the proof of the admissibility of the cut rule with the case for $\equiv$. (Global soundness)

**Theorem (Admissibility of the cut rule).** If $\Gamma \rightarrow A$ and $\Gamma, A \rightarrow C$, then $\Gamma \rightarrow C$.

**Proof.** By nested induction on the structure of: 1) cut formula $A$; 2) proof of $\Gamma \rightarrow A$; 3) proof of $\Gamma, A \rightarrow C$.

Case:

1) $A = A_1 \equiv A_2$.

2) the last inference rule in the proof $D$ of $\Gamma \rightarrow A$ is $\equiv R$.

3) the last inference rule in the proof $E$ of $\Gamma, A \rightarrow C$ is $\equiv L_1$.

4) $A$ is the principal formula of both $D$ and $E$.

First show the structure of proofs $D$ and $E$.

\[
D = \frac{\Gamma, A_1 \rightarrow A_2 \quad \Gamma, A_2 \rightarrow A_1}{\Gamma \rightarrow A_1 \equiv A_2} \quad E = \frac{\frac{\Gamma, A_1 \equiv A_2 \rightarrow A_1}{\Gamma, A_1 \equiv A_2} \quad \frac{\Gamma, A_1 \equiv A_2, A_2 \rightarrow C}{\Gamma, A_1 \equiv A_2 \rightarrow C}}{\equiv L_2}
\]

Then deduce $\Gamma \rightarrow C$. In each line, show the conclusion in the left side and its justification in the right side.

$E' :: \Gamma \rightarrow A_1$

$D' :: \Gamma, A_2 \rightarrow A_1 \equiv A_2$

$E'' :: \Gamma, A_2 \rightarrow C$

$D'' :: \Gamma \rightarrow A_2$

$\Gamma \rightarrow C$

by IH on $A_1 \equiv A_2$, $D$, $E_1$

by weakening $D :: \Gamma \rightarrow A_1 \equiv A_2$

by IH on $A_1 \equiv A_2$, $D'$, $E_2$

by IH on $A_1$, $E'$, $D_1$

by IH on $A_2$, $D''$, $E''$

3 Classical logic [15 pts]

We learned in class that constructive logic “degenerates” to classical logic if the axiom

\[
\frac{\neg \neg A \supset A \text{ true}}{DN}
\]

is added, where DN stands for ‘Double Negation.’ Another way to obtain classical logic is by adding either

\[
\frac{A \lor \neg A \text{ true}}{EM}
\]
or

\[(A \supset B) \supset A \] Pierce

where EM stands for ‘Excluded Middle.’

**Question 1. [10 pts]**

Prove that in the presence of the rule DN, the rule EM is derivable. Use the natural deduction system, but do not use hypothetical judgments. Note that \(A\) in the rule DN is a metavariable which can be instantiated to any proposition. You may write \(A\) instead of \(A \text{ true}\) in your proof, if it makes your proof more readable.

\[
\begin{align*}
\neg(A \lor \neg A) & \quad \text{true} \\
\frac{\text{DN}}{A \lor \neg A \quad \text{true}} \\
\frac{\text{I}^x}{\neg \neg (A \lor \neg A) \quad \text{true}} \\
\frac{\text{I}^y}{(A \supset B) \quad \text{true}} \\
\frac{\text{I}^z}{((A \supset B) \supset A) \quad \text{true}} \quad \text{Pierce}
\end{align*}
\]

**Question 2. [5 pts]**

Prove that in the presence of the rule EM, the rule Pierce is derivable. Use the natural deduction system, but do not use hypothetical judgments. You may write \(A\) instead of \(A \text{ true}\) in your proof, if it makes your proof more readable.

\[
\begin{align*}
\text{EM} \\
\frac{\text{I}^w}{((A \supset B) \supset A) \quad \text{true}} \\
\frac{\text{I}^v}{A \supset B \quad \text{true}} \\
\frac{\text{I}^u}{A \quad \text{true}} \\
\frac{\text{I}^x}{(A \supset B) \quad \text{true}} \\
\frac{\text{I}^y}{A \quad \text{true}} \\
\frac{\text{I}^z}{((A \supset B) \supset A) \quad \text{true}} \quad \text{Pierce}
\end{align*}
\]

**Question 3. [Extra credit]**

The rule DN allows us to conclude \(A \text{ true}\) whenever \(\neg \neg A \text{ true}\) is provable, or whenever an assumption of \(\neg A \text{ true}\) leads to a logical contradiction, as in typical proofs in math textbooks. By the rule EM, we only have to consider two possibilities \(A \text{ true}\) and \(\neg A \text{ true}\) when proving \(C \text{ true}\). Then how can we exploit the rule Pierce when proving \(A \text{ true}\)?

In order to prove \(A \text{ true}\), we only have to prove \((A \supset B) \supset A \text{ true}\). In order to prove \((A \supset B) \supset A \text{ true}\), we assume \(A \supset B \text{ true}\) and prove \(A \text{ true}\). Therefore, in order to prove \(A \text{ true}\), we may assume \(A \supset B \text{ true}\) for an arbitrary proposition \(B\) for free!

**Work sheet**