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CSE-490 Logic in Computer Science 2006
Midterm — Sample Solution

	Problem 1	Problem 2	Problem 3	Total
Score				
Max	50	35	15	100

1 Short questions [50 pts]

Answer each question below. All the questions assume constructive logic, not classical logic. For questions 1, 2, and 5, answer either true or false.

1.1 Propositional logic

Question 1. [5 pts]
$A \supset (B \vee C) \equiv (A \supset B) \vee (A \supset C)$ holds. True or false?
False.

Question 2. [5 pts]
$(A \vee B) \supset C \equiv (A \supset C) \wedge (B \supset C)$ holds. True or false?
True.

Question 3. [5 pts]
Give an example of a proof of $A\ \mathit{true}$ that is not normal, but does not contain a detour, either. Use the natural deduction system, but do **not** use hypothetical judgments. You may choose any proposition A you like.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vee A\ \mathit{true}}\,w \quad \dfrac{\dfrac{}{A\ \mathit{true}}\,x \quad \dfrac{}{A\ \mathit{true}}\,x}{A \wedge A\ \mathit{true}}\,\wedge\mathsf{I} \quad \dfrac{\dfrac{}{A\ \mathit{true}}\,y \quad \dfrac{}{A\ \mathit{true}}\,y}{A \wedge A\ \mathit{true}}\,\wedge\mathsf{I}}{A \wedge A\ \mathit{true}}\,\vee\mathsf{E}^{x,y}}{A\ \mathit{true}}\,\wedge\mathsf{E_L}}{(A \vee A) \supset A\ \mathit{true}}\,\supset\mathsf{I}^w$$

1.2 Proof terms

Question 4. [5 pts]
What is the type of the following proof term?

$$\lambda x\!:\!A.\, \mathsf{case}\ \mathsf{inl}_\bot\ x\ \mathsf{of}\ \mathsf{inl}\ y \Rightarrow (y, y) \mid \mathsf{inr}\ z \Rightarrow \mathsf{abort}_{A \wedge A}\ y$$

$A \supset (A \wedge A)$.

Question 5. [5 pts]
The following proof term represents a **long** normal proof where A and B are atomic propositions. True or false?

$$\lambda x\!:\!A \vee B.\, \mathsf{case}\ x\ \mathsf{of}\ \mathsf{inl}\ y \Rightarrow \mathsf{inr}_B\ y \mid \mathsf{inr}\ z \Rightarrow \mathsf{inl}_A\ z$$

True.

Question 6. [5 pts] Find two different proof terms that represent **long** normal proofs of the same judgment $A \supset (A \supset A)\ \mathit{true}$.
$\lambda x\!:\!A.\, \lambda y\!:\!A.\, x$ and $\lambda x\!:\!A.\, \lambda y\!:\!A.\, y$.

Question 7. [5 pts]
Apply β-reductions to reduce the following proof term to the simplest form:

$$(\lambda x{:}(A \wedge B) \vee C.\, \lambda y{:}(A \wedge B) \supset A.\, \lambda z{:}C \supset A.\, \mathsf{case}\ x\ \mathsf{of}\ \mathsf{inl}\ x_1 \Rightarrow y\ x_1 \mid \mathsf{inr}\ x_2 \Rightarrow z\ x_2)\ (\mathsf{inl}_C\ (M, N))\ (\lambda w{:}A \wedge B.\, \mathsf{fst}\ w)$$

$\lambda z{:}C \supset A.\, M.$

Question 8. [5 pts]
Apply β-reductions and **commuting conversions** to reduce to the following proof term to the simplest form:

$$\lambda w{:}A \vee A.\ \mathsf{fst}\ (\mathsf{case}\ (\mathsf{case}\ w\ \mathsf{of}\ \mathsf{inl}\ x \Rightarrow \mathsf{inl}_A\ x \mid \mathsf{inr}\ y \Rightarrow \mathsf{inr}_A\ y)\ \mathsf{of}\ \mathsf{inl}\ x' \Rightarrow (x', x') \mid \mathsf{inr}\ y' \Rightarrow (y', y'))$$

$\lambda w{:}A \vee A.\, \mathsf{case}\ w\ \mathsf{of}\ \mathsf{inl}\ x \Rightarrow x \mid \mathsf{inr}\ y \Rightarrow y.$

1.3 Sequent calculus

Question 9. [5 pts]
Give a proof of a sequent $\cdot \longrightarrow (\neg A \vee B) \supset (A \supset B)$. If not provable, state so.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \vee B, A, \neg A \longrightarrow A}\ \mathit{Init}}{\neg A \vee B, A, \neg A \longrightarrow B}\ \neg L \qquad \dfrac{}{\neg A \vee B, A, B \longrightarrow B}\ \mathit{Init}}{\dfrac{\neg A \vee B, A \longrightarrow B}{\neg A \vee B \longrightarrow A \supset B}\ {\supset}R}\ \vee L}{\cdot \longrightarrow (\neg A \vee B) \supset (A \supset B)}\ {\supset}R$$

Question 10. [5 pts]
Give a proof of a sequent $\cdot \longrightarrow (A \supset B) \supset (\neg A \vee B)$. If not provable, state so.
Not provable.

2 Logical equivalence [35 pts]

In the course notes, we use a notational definition of logical equivalence $\equiv$ given as follows:

$$A \equiv B \quad = \quad (A \supset B) \wedge (B \supset A)\ \mathit{true}$$

In this problem, we will define $\equiv$ as a logical connective, like $\supset$, $\wedge$, and $\vee$, so that $A \equiv B\ \mathit{true}$ holds if and only if $(A \supset B) \wedge (B \supset A)\ \mathit{true}$ holds. We extend the natural deduction system for propositional logic to incorporate $\equiv$ as a new logical connective orthogonal to the existing logical connectives, by providing its introduction and elimination rules. Thus we assume the following formation rule for $\equiv$:

$$\dfrac{A\ \mathit{prop} \quad B\ \mathit{prop}}{A \equiv B\ \mathit{prop}}\ {\equiv}\mathsf{F}$$

Question 1. [5 pts]

Propose an introduction rule $\equiv\mathsf{I}$ and two elimination rules $\equiv\mathsf{E}_1$ and $\equiv\mathsf{E}_2$ for $\equiv$. Do **<u>not</u>** use hypothetical judgments. Be careful not to destroy the orthogonality of the system.

$$\dfrac{\begin{array}{cc} \overline{A\ \mathit{true}}^{\,x} & \overline{B\ \mathit{true}}^{\,y} \\ \vdots & \vdots \\ B\ \mathit{true} & A\ \mathit{true} \end{array}}{A \equiv B\ \mathit{true}} \equiv\mathsf{I}^{x,y} \qquad \dfrac{A \equiv B\ \mathit{true} \quad A\ \mathit{true}}{B\ \mathit{true}} \equiv\mathsf{E}_1 \qquad \dfrac{A \equiv B\ \mathit{true} \quad B\ \mathit{true}}{A\ \mathit{true}} \equiv\mathsf{E}_2$$

Question 2. [5 pts]

Show a local reduction $\Longrightarrow_R$ for $\equiv$. (Local soundness)

$$\dfrac{\dfrac{\begin{array}{cc} \overline{A\ \mathit{true}}^{\,x} & \overline{B\ \mathit{true}}^{\,y} \\ \vdots & \vdots \\ B\ \mathit{true} & A\ \mathit{true} \end{array}}{A \equiv B\ \mathit{true}} \equiv\mathsf{I}^{x,y} \quad \begin{array}{c} \mathcal{D} \\ A\ \mathit{true} \end{array}}{B\ \mathit{true}} \equiv\mathsf{E} \qquad \Longrightarrow_R \qquad \begin{array}{c} \mathcal{D} \\ A\ \mathit{true} \\ \vdots \\ B\ \mathit{true} \end{array}$$

Question 3. [5 pts]

Show a local expansion $\Longrightarrow_E$ for $\equiv$. (Local completeness)

$$\begin{array}{c} \mathcal{D} \\ A \equiv B\ \mathit{true} \end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c} \mathcal{D} \\ A \equiv B\ \mathit{true} \end{array} \quad \overline{A\ \mathit{true}}^{\,x}}{B\ \mathit{true}} \equiv\mathsf{E}_1 \quad \dfrac{\begin{array}{c} \mathcal{D} \\ A \equiv B\ \mathit{true} \end{array} \quad \overline{B\ \mathit{true}}^{\,y}}{A\ \mathit{true}} \equiv\mathsf{E}_2}{A \equiv B\ \mathit{true}} \equiv\mathsf{I}^{x,y}$$

Question 4. [5 pts]

Rewrite the rules $\equiv\mathsf{I}$, $\equiv\mathsf{E}_1$, $\equiv\mathsf{E}_2$ for neutral and normal judgments by replacing $A\ \mathit{true}$ by $A\!\uparrow$ or $A\!\downarrow$. Call the resultant rules $\equiv\mathsf{I}\uparrow$, $\equiv\mathsf{E}\downarrow_1$, $\equiv\mathsf{E}\downarrow_2$.

$$\dfrac{\begin{array}{cc} \overline{A\!\downarrow}^{\,x} & \overline{B\!\downarrow}^{\,y} \\ \vdots & \vdots \\ B\!\uparrow & A\!\uparrow \end{array}}{A \equiv B\!\uparrow} \equiv\mathsf{I}\uparrow^{x,y} \qquad \dfrac{A \equiv B\!\downarrow \quad A\!\uparrow}{B\!\downarrow} \equiv\mathsf{E}\downarrow_1 \qquad \dfrac{A \equiv B\!\downarrow \quad B\!\uparrow}{A\!\downarrow} \equiv\mathsf{E}_2$$

Question 5. [5 pts]

Transform the rules $\equiv\mathsf{I}\uparrow$, $\equiv\mathsf{E}\downarrow_1$, $\equiv\mathsf{E}\downarrow_2$ to rules for sequent calculus. Call the resultant rules $\equiv\mathsf{R}$, $\equiv\mathsf{L}_1$, $\equiv\mathsf{L}_2$, respectively. For a collection of propositions in the left side of a sequent, you may use a metavariable Γ.

$$\dfrac{\Gamma, A \longrightarrow B \quad \Gamma, B \longrightarrow A}{\Gamma \longrightarrow A \equiv B} \equiv\mathsf{R}$$

$$\dfrac{\Gamma, A \equiv B \longrightarrow A \quad \Gamma, A \equiv B, B \longrightarrow C}{\Gamma, A \equiv B \longrightarrow C} \equiv\mathsf{L}_1 \qquad \dfrac{\Gamma, A \equiv B \longrightarrow B \quad \Gamma, A \equiv B, A \longrightarrow C}{\Gamma, A \equiv B \longrightarrow C} \equiv\mathsf{L}_1$$

Question 6. [5 pts]

Assume proofs $\overset{\mathcal{D}}{A \longrightarrow A}$ and $\overset{\mathcal{E}}{B \longrightarrow B}$ to prove $A \equiv B \longrightarrow A \equiv B$. You may refer to proofs of sequents obtained by weakening $A \longrightarrow A$ and $B \longrightarrow B$ as $\mathcal{D}$ and $\mathcal{E}$, respectively. (Global completeness)

$$\dfrac{\dfrac{\overset{\mathcal{D}}{A \equiv B, A \longrightarrow A} \quad \overset{\mathcal{E}}{A \equiv B, B \longrightarrow B}}{A \equiv B, A \longrightarrow B} \equiv\mathsf{L}_1 \quad \dfrac{\overset{\mathcal{E}}{A \equiv B, B \longrightarrow B} \quad \overset{\mathcal{D}}{A \equiv B, A \longrightarrow A}}{A \equiv B, B \longrightarrow A} \equiv\mathsf{L}_2}{A \equiv B \longrightarrow A \equiv B} \equiv\mathsf{R}$$

Question 7. [5 pts]

Extend the proof of the admissibility of the cut rule with the case for $\equiv$. (Global soundness)

Theorem (Admissiblity of the cut rule). *If* $\Gamma \longrightarrow A$ *and* $\Gamma, A \longrightarrow C$*, then* $\Gamma \longrightarrow C$*.*

Proof. By nested induction on the structure of: 1) cut formula A; 2) proof of $\Gamma \longrightarrow A$; 3) proof of $\Gamma, A \longrightarrow C$.

Case:

1) $A = A_1 \equiv A_2$.
2) the last inference rule in the proof $\mathcal{D}$ of $\Gamma \longrightarrow A$ is $\equiv\mathsf{R}$.
3) the last inference rule in the proof $\mathcal{E}$ of $\Gamma, A \longrightarrow C$ is $\equiv\mathsf{L}_1$.
4) A is the principal formula of both $\mathcal{D}$ and $\mathcal{E}$.

First show the structure of proofs $\mathcal{D}$ and $\mathcal{E}$.

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma, A_1 \longrightarrow A_2} \quad \overset{\mathcal{D}_2}{\Gamma, A_2 \longrightarrow A_1}}{\Gamma \longrightarrow A_1 \equiv A_2} \qquad \mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Gamma, A_1 \equiv A_2 \longrightarrow A_1} \quad \overset{\mathcal{E}_2}{\Gamma, A_1 \equiv A_2, A_2 \longrightarrow C}}{\Gamma, A_1 \equiv A_2 \longrightarrow C}$$

Then deduce $\Gamma \longrightarrow C$. In each line, show the conclusion in the left side and its justification in the right side.

$\mathcal{E}' :: \Gamma \longrightarrow A_1$	by IH on $A_1 \equiv A_2$, $\mathcal{D}$, $\mathcal{E}_1$
$\mathcal{D}' :: \Gamma, A_2 \longrightarrow A_1 \equiv A_2$	by weakening $\mathcal{D} :: \Gamma \longrightarrow A_1 \equiv A_2$
$\mathcal{E}'' :: \Gamma, A_2 \longrightarrow C$	by IH on $A_1 \equiv A_2$, $\mathcal{D}'$, $\mathcal{E}_2$
$\mathcal{D}'' :: \Gamma \longrightarrow A_2$	by IH on A_1, $\mathcal{E}'$, $\mathcal{D}_1$
$\Gamma \longrightarrow C$	by IH on A_2, $\mathcal{D}''$, $\mathcal{E}''$

3 Classical logic [15 pts]

We learned in class that constructive logic "degenerates" to classical logic if the axiom

$$\dfrac{}{\neg\neg A \supset A\ \mathit{true}}\ \mathsf{DN}$$

is added, where DN stands for 'Double Negation.' Another way to obtain classical logic is by adding either

$$\dfrac{}{A \vee \neg A\ \mathit{true}}\ \mathsf{EM}$$

or

$$\frac{}{((A \supset B) \supset A) \supset A\ \mathit{true}}\ \mathsf{Pierce}$$

where EM stands for 'Excluded Middle.'

Question 1. [10 pts]

Prove that in the presence of the rule DN, the rule EM is derivable. Use the natural deduction system, but do **not** use hypothetical judgments. Note that A in the rule DN is a metavariable which can be instantiated to any proposition. You may write A instead of $A\ \mathit{true}$ in your proof, if it makes your proof more readable.

$$\dfrac{\dfrac{}{\neg\neg(A \vee \neg A) \supset (A \vee \neg A)\ \mathit{true}}\ \mathsf{DN} \qquad \dfrac{\dfrac{\dfrac{}{\neg(A \vee \neg A)\ \mathit{true}}^{x} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A \vee \neg A)\ \mathit{true}}^{x} \qquad \dfrac{\dfrac{}{A\ \mathit{true}}^{y}}{A \vee \neg A\ \mathit{true}}\ \vee\mathsf{I_L}}{\bot\ \mathit{true}}\ \supset\mathsf{E}}{\neg A\ \mathit{true}}\ \supset\mathsf{I}^{y}}{A \vee \neg A\ \mathit{true}}\ \vee\mathsf{I_R}}{\bot}\ \supset\mathsf{E}}{\neg\neg(A \vee \neg A)\ \mathit{true}}\ \supset\mathsf{I}^{x}}{A \vee \neg A\ \mathit{true}}\ \supset\mathsf{E}$$

Question 2. [5 pts]

Prove that in the presence of the rule EM, the rule Pierce is derivable. Use the natural deduction system, but do **not** use hypothetical judgments. You may write A instead of $A\ \mathit{true}$ in your proof, if it makes your proof more readable.

$$\dfrac{\dfrac{}{A \vee \neg A\ \mathit{true}}\ \mathsf{EM} \qquad \dfrac{\dfrac{}{A\ \mathit{true}}^{x}}{((A \supset B) \supset A) \supset A\ \mathit{true}}\ \supset\mathsf{I}^{w} \qquad \dfrac{\dfrac{\dfrac{}{(A \supset B) \supset A\ \mathit{true}}^{z} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg A\ \mathit{true}}^{y} \qquad \dfrac{}{A\ \mathit{true}}^{w}}{\bot\ \mathit{true}}\ \supset\mathsf{E}}{B\ \mathit{true}}\ \bot\mathsf{E}}{A \supset B\ \mathit{true}}\ \supset\mathsf{I}^{w}}{A\ \mathit{true}}\ \supset\mathsf{E}}{((A \supset B) \supset A) \supset A\ \mathit{true}}\ \supset\mathsf{I}^{z}}{((A \supset B) \supset A) \supset A\ \mathit{true}}\ \vee\mathsf{E}^{x,y}$$

Question 3. [Extra credit]

The rule DN allows us to conclude $A\ \mathit{true}$ whenever $\neg\neg A\ \mathit{true}$ is provable, or whenever an assumption of $\neg A\ \mathit{true}$ leads to a logical contradiction, as in typical proofs in math textbooks. By the rule EM, we only have to consider two possibilities $A\ \mathit{true}$ and $\neg A\ \mathit{true}$ when proving $C\ \mathit{true}$. Then how can we exploit the rule Pierce when proving $A\ \mathit{true}$?

In order to prove $A\ \mathit{true}$, we only have to prove $(A \supset B) \supset A\ \mathit{true}$. In order to prove $(A \supset B) \supset A\ \mathit{true}$, we assume $A \supset B\ \mathit{true}$ and prove $A\ \mathit{true}$. Therefore, in order to prove $A\ \mathit{true}$, we may assume $A \supset B\ \mathit{true}$ for an arbitrary proposition B for free!

Work sheet