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# CSE-433 Logic in Computer Science 2007
# Final exam — Sample Solution

- This is a closed-book exam. No other material is permitted.
- It consists of 4 problems worth a total of 175 points.
- There are 12 pages in this exam, including 3 work sheets.
- Try to use work sheets before writing your answers. Write your answers clearly and legibly.
- You have 3 hours for this exam.

| | Problem 1 | Problem 2 | Problem 3 | Problem 4 | Total |
|---|---|---|---|---|---|
| Score | | | | | |
| Max | 30 | 55 | 50 | 40 | 175 |

# 1 Admissibility of Harrop's rule [30 pts]

In constructive logic, there are some admissible rules that are not derivable. In the midterm exam, we considered Harrop's rule as an example of such a rule and showed that it is not derivable. As we said before, in this problem, we will show that Harrop's rule is indeed admissible. As a reminder, Harrop's rule is as follows:

$$\frac{\neg A \supset (B \vee C)\ \mathit{true}}{(\neg A \supset B) \vee (\neg A \supset C)\ \mathit{true}}\ \mathit{Harrop}$$

where $A, B$, and $C$ are all propositions.

You will need to use the following inference rules for neutral and normal judgments.

$$\frac{\begin{array}{c}\overline{A\downarrow}^{\,x}\\ \vdots \\ B\uparrow\end{array}}{A \supset B\uparrow}\ \supset\mathsf{I}^x_\uparrow \qquad \frac{A \supset B\downarrow \quad A\uparrow}{B\downarrow}\ \supset\mathsf{E}_\downarrow \qquad \frac{A\uparrow \quad B\uparrow}{A \wedge B\uparrow}\ \wedge\mathsf{I}_\uparrow \qquad \frac{A \wedge B\downarrow}{A\downarrow}\ \wedge\mathsf{E}_{\mathsf{L}\downarrow} \qquad \frac{A \wedge B\downarrow}{B\downarrow}\ \wedge\mathsf{E}_{\mathsf{R}\downarrow}$$

$$\frac{A\uparrow}{A \vee B\uparrow}\ \vee\mathsf{I}_{\mathsf{L}\uparrow} \qquad \frac{B\uparrow}{A \vee B\uparrow}\ \vee\mathsf{I}_{\mathsf{R}\uparrow} \qquad \frac{A \vee B\downarrow \quad \begin{array}{c}\overline{A\downarrow}^{\,x}\\ \vdots \\ C\uparrow\end{array} \quad \begin{array}{c}\overline{B\downarrow}^{\,y}\\ \vdots \\ C\uparrow\end{array}}{C\uparrow}\ \vee\mathsf{E}^{x,y}_\uparrow$$

$$\frac{}{\top\uparrow}\ \top\mathsf{I}_\uparrow \qquad \frac{\bot\downarrow}{C\uparrow}\ \bot\mathsf{E}_\uparrow \qquad \frac{A\downarrow}{A\uparrow}\ \downarrow\uparrow \qquad \frac{\begin{array}{c}\overline{A\downarrow}^{\,x}\\ \vdots \\ \bot\uparrow\end{array}}{\neg A\uparrow}\ \neg\mathsf{I}_\uparrow{}^x \qquad \frac{\neg A\downarrow \quad A\uparrow}{\bot\downarrow}\ \neg\mathsf{E}_\downarrow$$

- You should annotate every part of your derivation with the name of the inference rule (*e.g.*, $\supset\mathsf{E}_\downarrow$) and also a label if applicable (*e.g.*, $\supset\mathsf{I}^x_\uparrow$). In particular, you should annotate each hypothesis with some variable (*e.g.*, $\overline{A\downarrow}^{\,x}$).

Since we cannot directly show that Harrop's rule is admissible, we need to analyze the proof structure of the premise, $\neg A \supset (B \vee C)\ \mathit{true}$, to show that $(\neg A \supset B) \vee (\neg A \supset C)\ \mathit{true}$ is provable. Recall that in constructive logic, $A\ \mathit{true}$ is provable if and only if $A\uparrow$ is provable. Therefore we will show that Harrop's rule is admissible by showing that $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$ is provable in the presence of a proof of $\neg A \supset (B \vee C)\uparrow$.

*Proof.* The proof of $\neg A \supset (B \vee C)\uparrow$, if any, must be of the following form:

$$\frac{\mathcal{D}\left\{\begin{array}{c}\overline{\neg A\downarrow}^{\,x}\\ \vdots \\ B \vee C\uparrow\end{array}\right.}{\neg A \supset (B \vee C)\uparrow}\ \supset\mathsf{I}^x_\uparrow$$

We need to show that $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$ is provable by exploring *all* possible cases of the structure of $\mathcal{D}$.

*Case* 1. Suppose that in $\mathcal{D}$, the rule $\neg\mathsf{E}_{\downarrow}$ is applied to $\overline{\neg A\downarrow}^{x}$. Then there must be a proof of $A\uparrow$ under the hypothesis $\overline{\neg A\downarrow}^{x}$. Let this proof be $\mathcal{E}$. Use $\mathcal{E}$ to build a proof of $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$ (10 pts):

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg A\downarrow}^{x} \qquad \left.\begin{matrix}\overline{\neg A\downarrow}^{x}\\ \vdots \\ A\uparrow\end{matrix}\right\}\mathcal{E}}{\bot\downarrow}\ \neg\mathsf{E}_{\downarrow}}{B\uparrow}\ \bot\mathsf{E}_{\uparrow}}{\neg A \supset B\uparrow}\ \supset\mathsf{I}^{x}_{\uparrow}}{(\neg A \supset B) \vee (\neg A \supset C)\uparrow}\ \vee\mathsf{I}_{\mathsf{L}\uparrow}$$

*Case* 2. Suppose that in $\mathcal{D}$, the rule $\vee\mathsf{I}_{\mathsf{L}\uparrow}$ is applied to deduce $B \vee C\uparrow$. Then whtere must be a proof of $B\uparrow$ under the hypothesis $\overline{\neg A\downarrow}^{x}$. Let this proof be $\mathcal{E}$. Use $\mathcal{E}$ to build a proof of $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$ (10 pts):

$$\dfrac{\dfrac{\mathcal{E}\left\{\begin{matrix}\overline{\neg A\downarrow}^{x}\\ \vdots \\ B\uparrow\end{matrix}\right.}{\neg A \supset B\uparrow}\ \supset\mathsf{I}^{x}_{\uparrow}}{(\neg A \supset B) \vee (\neg A \supset C)\uparrow}\ \vee\mathsf{I}_{\mathsf{L}\uparrow}$$

*Case* 3. Suppose that in $\mathcal{D}$, the rule $\vee\mathsf{I}_{\mathsf{R}\uparrow}$ is applied to deduce $B \vee C\uparrow$. Then there must be a proof of $C\uparrow$ under the hypothesis $\overline{\neg A\downarrow}^{x}$. Let this proof be $\mathcal{E}$. Use $\mathcal{E}$ to build a proof of $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$ (10 pts):

$$\dfrac{\dfrac{\mathcal{E}\left\{\begin{matrix}\overline{\neg A\downarrow}^{x}\\ \vdots \\ C\uparrow\end{matrix}\right.}{\neg A \supset C\uparrow}\ \supset\mathsf{I}^{x}_{\uparrow}}{(\neg A \supset B) \vee (\neg A \supset C)\uparrow}\ \vee\mathsf{I}_{\mathsf{R}\uparrow}$$

The above three cases are the only possible cases of the structure of $\mathcal{D}$. Therefore in the presence of a proof of $\neg A \supset (B \vee C)\uparrow$, we can always prove $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$, which means that Harrop's rule is admissible.

## 2 Datatypes [55 pts]

We use the following definition of proof terms for the predicate $m < n$:

$$\text{proof term} \quad M \quad ::= \quad \cdots \mid \mathsf{ltI}_0 \mid \mathsf{ltI}_\mathsf{s}(M) \mid \mathsf{ltE}_0(M) \mid \mathsf{ltE}_\mathsf{s}(M)$$

$$\frac{}{\mathsf{ltI}_0 : \mathbf{0} < \mathbf{s}(n)} <\mathsf{I}_0 \qquad \frac{M : m < n}{\mathsf{ltI}_\mathsf{s}(M) : \mathbf{s}(m) < \mathbf{s}(n)} <\mathsf{I}_\mathsf{s} \qquad \frac{M : m < \mathbf{0}}{\mathsf{ltE}_0(M) : C} <\mathsf{E}_0 \qquad \frac{M : \mathbf{s}(m) < \mathbf{s}(n)}{\mathsf{ltE}_\mathsf{s}(M) : m < n} <\mathsf{E}_\mathsf{s}$$

We use the following definition of proof terms for the predicate $m =_\mathsf{N} n$:

$$\text{proof term} \quad M \quad ::= \quad \cdots \mid \mathsf{eqI}_0 \mid \mathsf{eqI}_\mathsf{s}(M) \mid \mathsf{eqE}_{0\mathsf{s}}(M) \mid \mathsf{eqE}_{\mathsf{s}0}(M) \mid \mathsf{eqE}_\mathsf{s}(M)$$

$$\frac{}{\mathsf{eqI}_0 : \mathbf{0} =_\mathsf{N} \mathbf{0}} =_\mathsf{N}\mathsf{I}_0 \qquad \frac{M : m =_\mathsf{N} n}{\mathsf{eqI}_\mathsf{s}(M) : \mathbf{s}(m) =_\mathsf{N} \mathbf{s}(n)} =_\mathsf{N}\mathsf{I}_\mathsf{s}$$

$$\frac{M : \mathbf{0} =_\mathsf{N} \mathbf{s}(n)}{\mathsf{eqE}_{0\mathsf{s}}(M) : C} =_\mathsf{N}\mathsf{E}_{0\mathsf{s}} \qquad \frac{M : \mathbf{s}(m) =_\mathsf{N} \mathbf{0}}{\mathsf{eqE}_{\mathsf{s}0}(M) : C} =_\mathsf{N}\mathsf{E}_{\mathsf{s}0} \qquad \frac{M : \mathbf{s}(m) =_\mathsf{N} \mathbf{s}(n)}{\mathsf{eqE}_\mathsf{s}(M) : m =_\mathsf{N} n} =_\mathsf{N}\mathsf{E}_\mathsf{s}$$

The following proof term represents a proof by induction on natural numbers:

$$\text{proof term} \quad M \quad ::= \quad \cdots \mid \mathsf{ind}\ u(t)\ \mathsf{of}\ u(\mathbf{0}) \Rightarrow M \mid u(\mathbf{s}(x)) \Rightarrow N$$

**Question 1. [5 pts]** Define a function *append* concatenating two lists:

$$\begin{array}{rcl} \mathit{append}\ \mathbf{nil}^\tau\ t & = & t \\ \mathit{append}\ (x :: l)\ t & = & x :: (\mathit{append}\ l\ t) \end{array}$$

$$\begin{array}{l} \mathit{append} \in \mathsf{list}\ \tau \to \mathsf{list}\ \tau \to \mathsf{list}\ \tau \\ \mathit{append}\ =\ \lambda y \in \mathsf{list}\ \tau.\, \lambda z \in \mathsf{list}\ \tau.\, \mathbf{rec}\ f(y)\ \mathbf{of}\ f(\mathbf{nil}) \Rightarrow z \mid f(x :: l) \Rightarrow x :: f(l) \end{array}$$

**Question 2. [5 pts]** Give a proof term of type $(\exists x \in \tau. A(x) \vee B(x)) \supset ((\exists x \in \tau. A(x)) \vee (\exists x \in \tau. B(x)))$. You may omit type annotations in injection terms. For example, you may write $\mathsf{inl}\ M$ when $\mathsf{inl}_A\ M$ is expected.

$$\lambda z{:}\exists x \in \tau. A(x) \vee B(x).\, \mathsf{let}\ \langle x, w\rangle = z\ \mathsf{in}\ \mathsf{case}\ w\ \mathsf{of}\ \mathsf{inl}\ y_1.\, \mathsf{inl}_{\exists x \in \tau. B(x)}\ \langle x, y_1\rangle \mid \mathsf{inr}\ y_2.\, \mathsf{inr}_{\exists x \in \tau. A(x)}\ \langle x, y_2\rangle.$$

**Question 3. [5 pts]** The specification of a proof term *pred* of type $\forall x \in \mathsf{nat}. \neg(x =_\mathsf{N} \mathbf{0}) \supset \exists y \in \mathsf{nat}. \mathbf{s}(y) =_\mathsf{N} x$ is:

$$\begin{array}{llllll} & x & v : \neg(x =_\mathsf{N} \mathbf{0}) & & \text{proof term of type } \exists y \in \mathsf{nat}. \mathbf{s}(y) =_\mathsf{N} x \\ \mathit{pred} & \mathbf{0} & v : \neg(\mathbf{0} =_\mathsf{N} \mathbf{0}) & = & \mathsf{abort}_{\exists y \in \mathsf{nat}. \mathbf{s}(y) =_\mathsf{N} \mathbf{0}}\ (v\ \mathsf{eqI}_0) \\ \mathit{pred} & \mathbf{s}(x') & v : \neg(\mathbf{s}(x') =_\mathsf{N} \mathbf{0}) & = & \langle x', \mathsf{eqI}_\mathsf{s}(\mathit{eqNat}\ x')\rangle \end{array}$$

Here *eqNat* is defined as $\lambda x \in \mathsf{nat}.\, \mathsf{ind}\ u(x)\ \mathsf{of}\ u(\mathbf{0}) \Rightarrow \mathsf{eqI}_0 \mid u(\mathbf{s}(x')) \Rightarrow \mathsf{eqI}_\mathsf{s}(u(x'))$. Give a definition of *pred* based on this specification.

$$\mathit{pred}\ =\ \lambda x \in \mathsf{nat}.\, \mathsf{case}\ x\ \mathsf{of}\ \left\{ \begin{array}{l} \mathbf{0} \Rightarrow \lambda v{:}\neg(\mathbf{0} =_\mathsf{N} \mathbf{0}).\, \mathsf{abort}_{\exists y \in \mathsf{nat}. \mathbf{s}(y) =_\mathsf{N} \mathbf{0}}\ (v\ \mathsf{eqI}_0) \\ \mathbf{s}(x') \Rightarrow \lambda v{:}\neg(\mathbf{s}(x') =_\mathsf{N} \mathbf{0}).\, \langle x', \mathsf{eqI}_\mathsf{s}(\mathit{eqNat}\ x')\rangle \end{array} \right.$$

**Question 4. [15 pts]** We wish to find a proof term *ltSucc* of type

$$\forall m \in \mathsf{nat}. \forall n \in \mathsf{nat}. m < \mathbf{s}(n) \supset (m =_\mathsf{N} n \vee m < n)$$

(which you used in Assignment 8). Write a specification of $ltSucc$ (similarly to $pred$) and then translate it to a definition as in Question 3.

Specification:

$$
\begin{array}{lllllll}
 & m & n & v : m < \mathbf{s}(n) & & \text{proof term of type } m =_{\mathsf{N}} n \vee m < n \\
ltSucc & \mathbf{0} & \mathbf{0} & v : \mathbf{0} < \mathbf{s}(\mathbf{0}) & = & \mathsf{inl}_{\mathbf{0}<\mathbf{0}}\ \mathsf{eqI}_0 \\
ltSucc & \mathbf{0} & \mathbf{s}(n') & v : \mathbf{0} < \mathbf{s}(\mathbf{s}(n')) & = & \mathsf{inr}_{\mathbf{0}=_{\mathsf{N}}\mathbf{s}(n')}\ \mathsf{ltI}_0 \\
ltSucc & \mathbf{s}(m') & \mathbf{0} & v : \mathbf{s}(m') < \mathbf{s}(\mathbf{0}) & = & \mathsf{ltE}_0(\mathsf{ltE}_{\mathsf{s}}(v)) \\
ltSucc & \mathbf{s}(m') & \mathbf{s}(n') & v : \mathbf{s}(m') < \mathbf{s}(\mathbf{s}(n')) & = & \mathsf{case}\ ltSucc\ m'\ n'\ \mathsf{ltE}_{\mathsf{s}}(v)\ \mathsf{of}\ \left\{ \begin{array}{l} \ \ \mathsf{inl}\ x.\ \mathsf{inl}_{\mathbf{s}(m')<\mathbf{s}(n')}\ \mathsf{eqI}_{\mathsf{s}}(x) \\ \mid\ \mathsf{inr}\ y.\ \mathsf{inr}_{\mathbf{s}(m')=_{\mathsf{N}}\mathbf{s}(n')}\ \mathsf{ltI}_{\mathsf{s}}(y) \end{array} \right.
\end{array}
$$

Definition:

$$
\lambda m \in \mathsf{nat}.\ \mathsf{ind}\ u(m)\ \mathsf{of} \left\{ \begin{array}{l}
u(\mathbf{0}) \Rightarrow \lambda n \in \mathsf{nat}.\ \mathsf{case}\ n\ \mathsf{of} \left\{ \begin{array}{l} \mathbf{0} \Rightarrow \lambda v : \mathbf{0} < \mathbf{s}(\mathbf{0}).\ \mathsf{inl}_{\mathbf{0}<\mathbf{0}}\ \mathsf{eqI}_0 \\ \mathbf{s}(n') \Rightarrow \lambda v : \mathbf{0} < \mathbf{s}(\mathbf{s}(n')).\ \mathsf{inr}_{\mathbf{0}=_{\mathsf{N}}\mathbf{s}(n')}\ \mathsf{ltI}_0 \end{array} \right. \\
u(\mathbf{s}(m')) \Rightarrow \lambda n \in \mathsf{nat}.\ \mathsf{case}\ n\ \mathsf{of} \left\{ \begin{array}{l} \mathbf{0} \Rightarrow \lambda v : \mathbf{s}(m') < \mathbf{s}(\mathbf{0}).\ \mathsf{ltE}_0(\mathsf{ltE}_{\mathsf{s}}(v)) \\ \mathbf{s}(n') \Rightarrow \lambda v : \mathbf{s}(m') < \mathbf{s}(\mathbf{s}(n')). \end{array} \right. \\
\qquad\qquad \mathsf{case}\ u(m')\ n'\ \mathsf{ltE}_{\mathsf{s}}(v)\ \mathsf{of} \left\{ \begin{array}{l} \ \ \mathsf{inl}\ x.\ \mathsf{inl}_{\mathbf{s}(m')<\mathbf{s}(n')}\ \mathsf{eqI}_{\mathsf{s}}(x) \\ \mid\ \mathsf{inr}\ y.\ \mathsf{inr}_{\mathbf{s}(m')=_{\mathsf{N}}\mathbf{s}(n')}\ \mathsf{ltI}_{\mathsf{s}}(y) \end{array} \right.
\end{array} \right.
$$

**Question 5. [5 pts]** Write the elimination rule for the predicate $m =_{\mathsf{N}} n$ based on induction on predicates (which is given in the Course Notes):

$$
\dfrac{m_0 =_{\mathsf{N}} n_0\ true \qquad A(\mathbf{0},\mathbf{0})\ true \qquad \begin{array}{c} \overline{m \in \mathsf{nat}} \quad \overline{n \in \mathsf{nat}} \quad \overline{m =_{\mathsf{N}} n}^{\,w} \quad \overline{A(m,n)\ true}^{\,u(m,n)} \\ \vdots \\ A(\mathbf{s}(m),\mathbf{s}(n))\ true \end{array}}{A(m_0,n_0)\ true} =_{\mathsf{N}}\mathsf{E}_I^{w,u(m,n)}
$$

**Question 6. [5 pts]** Use this elimination rule to prove $\forall x \in \mathsf{nat}.\forall y \in \mathsf{nat}.x =_{\mathsf{N}} y \supset y =_{\mathsf{N}} x\ true$:

$$
\dfrac{\dfrac{\dfrac{\overline{x =_{\mathsf{N}} y\ true}^{\,w} \qquad \dfrac{}{\mathbf{0} =_{\mathsf{N}} \mathbf{0}\ true} =_{\mathsf{N}}\mathsf{I}_0 \qquad \dfrac{\overline{n =_{\mathsf{N}} m\ true}^{\,u(m,n)}}{\mathbf{s}(n) =_{\mathsf{N}} \mathbf{s}(m)\ true} =_{\mathsf{N}}\mathsf{I}_{\mathsf{s}}}{y =_{\mathsf{N}} x\ true} =_{\mathsf{N}}\mathsf{E}_I^{w,u(m,n)}}{x =_{\mathsf{N}} y \supset y =_{\mathsf{N}} x\ true} \supset\mathsf{I}^w}{\dfrac{\forall y \in \mathsf{nat}.x =_{\mathsf{N}} y \supset y =_{\mathsf{N}} x\ true}{\forall x \in \mathsf{nat}.\forall y \in \mathsf{nat}.x =_{\mathsf{N}} y \supset y =_{\mathsf{N}} x\ true} \forall\mathsf{I}} \forall\mathsf{I}
$$

**Question 7. [5 pts]** Assuming definitional equality, give a proof term of type $\forall x \in \mathsf{nat}.\forall y \in \mathsf{nat}.x + \mathbf{s}(y) =_{\mathsf{N}} \mathbf{s}(x + y)$:

$$
\lambda x \in \mathsf{nat}.\ \lambda y \in \mathsf{nat}.\ \mathsf{ind}\ u(x)\ \mathsf{of} \left\{ \begin{array}{l} u(\mathbf{0}) \Rightarrow eqNat\ \mathbf{s}(y) \\ u(\mathbf{s}(x')) \Rightarrow \mathsf{eqI}_{\mathsf{s}}(u(x')) \end{array} \right.
$$

**Question 8. [5 pts]** The following inference rule is derivable. True or false?

$$
\dfrac{A(t)\ true \qquad t =_{\mathsf{N}} s}{A(s)\ true}\ NatEq\mathsf{E}
$$

False

**Question 9. [5 pts]** The rule $NatEq\mathsf{E}$ in the previous question is admissible. True or false?
True

## 3 Classical logic [50 pts]

In this problem, we use contexts $\Gamma$ and $\Delta$ defined as follows:

$$\begin{array}{rcl} \Gamma & ::= & \cdot \mid \Gamma, x : A \\ \Delta & ::= & \cdot \mid \Delta, x : A \ \mathit{false} \end{array}$$

We use $\Gamma; \Delta \vdash_{\mathsf{K}} C \ \mathit{true}$ for typechecking proof terms in classical logic and $\Gamma \vdash_{\mathsf{I}} M : C$ for typechecking proof terms in constructive logic.

Here are the rules Contra↑ and Contra↓ given in the Course Notes:

$$\frac{\Gamma; \Delta, A \ \mathit{false} \vdash_{\mathsf{K}} A \ \mathit{true}}{\Gamma; \Delta \vdash_{\mathsf{K}} A \ \mathit{true}}\ \mathsf{Contra}\!\uparrow \qquad \frac{\Gamma; \Delta, A \ \mathit{false} \vdash_{\mathsf{K}} A \ \mathit{true}}{\Gamma; \Delta, A \ \mathit{false} \vdash_{\mathsf{K}} C \ \mathit{true}}\ \mathsf{Contra}\!\downarrow$$

**Question 1. [5 pts]** Use the rules Contra↑ and Contra↓ to show that the rule Peirce is derivable.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \supset B) \supset A \ \mathit{true}; A \ \mathit{false} \vdash_{\mathsf{K}} (A \supset B) \supset A \ \mathit{true}}\ \mathsf{Hyp} \quad \dfrac{\dfrac{\dfrac{}{(A \supset B) \supset A \ \mathit{true}, A \ \mathit{true}; A \ \mathit{false} \vdash_{\mathsf{K}} A \ \mathit{true}}\ \mathsf{Hyp}}{(A \supset B) \supset A \ \mathit{true}, A \ \mathit{true}; A \ \mathit{false} \vdash_{\mathsf{K}} B \ \mathit{true}}\ \mathsf{Contra}\!\downarrow}{(A \supset B) \supset A \ \mathit{true}; A \ \mathit{false} \vdash_{\mathsf{K}} A \supset B \ \mathit{true}}\ {\supset}\mathsf{I}}{(A \supset B) \supset A \ \mathit{true}; A \ \mathit{false} \vdash_{\mathsf{K}} A \ \mathit{true}}\ {\supset}\mathsf{E}}{(A \supset B) \supset A \ \mathit{true}; \cdot \vdash_{\mathsf{K}} A \ \mathit{true}}\ \mathsf{Contra}\!\uparrow}{\cdot; \cdot \vdash_{\mathsf{K}} ((A \supset B) \supset A) \supset A \ \mathit{true}}\ {\supset}\mathsf{I}$$

**Question 2. [20 pts]** We use the following double-negation translation for the fragment of propositional logic with implication:

$$\begin{array}{rcl} P^{\circ} & = & P \\ (A \supset B)^{\circ} & = & A^{\circ} \supset \neg\neg B^{\circ} \end{array}$$

For each case below, complete the CPS translation $M^{\circ}$ of a given proof term $M$. Your translation should satisfy the invariant specified in the following theorem:

**Theorem.** *If $\Gamma; \Delta \vdash_{\mathsf{K}} M : C$, there exists a proof term $M^{\circ}$ such that $\Gamma^{\circ}, \neg\Delta^{\circ} \vdash_{\mathsf{I}} M^{\circ} : \neg\neg C^{\circ}$ where*

$$\begin{array}{rcl} \Gamma^{\circ} & = & \{x : A^{\circ} \mid x : A \in \Gamma\} \\ \neg\Delta^{\circ} & = & \{x : \neg A^{\circ} \mid x : A \ \mathit{false} \in \Delta\}. \end{array}$$

Case $\dfrac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathsf{K}} x : A}\ \mathsf{Hyp}$

$x^{\circ} = \lambda k\!:\!A^{\circ} \supset \bot.\ k\ x$

Case $\dfrac{\Gamma, x : A; \Delta \vdash_{\mathsf{K}} M : B}{\Gamma; \Delta \vdash_{\mathsf{K}} \lambda x\!:\!A.\ M : A \supset B}\ {\supset}\mathsf{I}$

$(\lambda x\!:\!A.\ M)^{\circ} = \lambda k\!:\!(A^{\circ} \supset \neg\neg B^{\circ}) \supset \bot.\ k\ (\lambda x\!:\!A^{\circ}.\ M^{\circ})$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathsf{K}} N : A}{\Gamma; \Delta \vdash_{\mathsf{K}} M\ N : B} \supset\mathsf{E}$

$(M\ N)^\circ = \lambda k\!:\!B^\circ \supset \bot.\, M^\circ\ (\lambda x\!:\!A^\circ \supset \neg\neg B^\circ.\, N^\circ\ (\lambda y\!:\!A^\circ.\, x\ y\ k))$

Case $\dfrac{\Gamma; \Delta, x : A\ \mathit{false} \vdash_{\mathsf{K}} M : A}{\Gamma; \Delta \vdash_{\mathsf{K}} \mathsf{callcc}\ x : A\ \mathit{false}.\, M : A}$ Callcc

$(\mathsf{callcc}\ x : A\ \mathit{false}.\, M)^\circ = \lambda k\!:\!A^\circ \supset \bot.\, [k/x]M^\circ\ k$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \quad x : A\ \mathit{false} \in \Delta}{\Gamma; \Delta \vdash_{\mathsf{K}} \mathsf{throw}\ M\ \mathsf{to}\ x : C}$ Throw

$(\mathsf{throw}\ M\ \mathsf{to}\ x)^\circ = \lambda k\!:\!C^\circ \supset \bot.\, M^\circ\ x$

**Question 3. [25 pts]** The above CPS transformation corresponds to the call-by-value reduction strategy. To be specific, the translation of $M\ N$ specifies that we finish evaluating $N$ before we apply the function from $M$ to the result of evaluating $N$.

In this question, we will develop a variant of the CPS translation that corresponds to the call-by-name reduction strategy. We use the following double-negation translation (known as *Kolmogorov double negation translation*) in which $(A \supset B)^\circ$ places $\neg\neg$ before both $A^\circ$ and $B^\circ$:

$$\begin{array}{rcl} P^\circ & = & P \\ (A \supset B)^\circ & = & \neg\neg A^\circ \supset \neg\neg B^\circ \end{array}$$

For each case below, complete the CPS translation $M^\circ$ of a given proof term $M$. Your translation should satisfy the invariant specified in the following theorem:

**Theorem.** *If $\Gamma; \Delta \vdash_{\mathsf{K}} M : C$, there exists a proof term $M^\circ$ such that $\Gamma^\circ, \neg\Delta^\circ \vdash_{\mathsf{I}} M^\circ : \neg\neg C^\circ$ where*

$$\begin{array}{rcl} \Gamma^\circ & = & \{x : \neg\neg A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ & = & \{x : \neg A^\circ \mid x : A\ \mathit{false} \in \Delta\}. \end{array}$$

Note the change in the definition of $\Gamma^\circ$ which now assigns to $x$ type $\neg\neg A^\circ$. The translation of $\mathsf{callcc}\ x : A\ \mathit{false}.\, M$ and $\mathsf{throw}\ M\ \mathsf{to}\ x$ is the same as in the previous CPS translation and is omitted.

Case $\dfrac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathsf{K}} x : A}$ Hyp

$x^\circ = \lambda k\!:\!A^\circ \supset \bot.\, x\; k$

Case $\dfrac{\Gamma, x : A; \Delta \vdash_{\mathsf{K}} M : B}{\Gamma; \Delta \vdash_{\mathsf{K}} \lambda x\!:\!A.\, M : A \supset B} \supset\mathsf{I}$

$(\lambda x\!:\!A.\, M)^\circ = \lambda k\!:\!(\neg\neg A^\circ \supset \neg\neg B^\circ) \supset \bot.\, k\; (\lambda x\!:\!\neg\neg A^\circ.\, M^\circ)$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathsf{K}} N : A}{\Gamma; \Delta \vdash_{\mathsf{K}} M\; N : B} \supset\mathsf{E}$

$(M\; N)^\circ = \lambda k\!:\!B^\circ \supset \bot.\, M^\circ\; (\lambda x\!:\!\neg\neg A^\circ \supset \neg\neg B^\circ.\, x\; N^\circ\; k)$

# 4 Linear logic [40 pts]

For each proposition $A$ in linear logic below, prove its truth using natural deduction with the linear hypothetical judgment $\Delta \Vdash A$. You should annotate every part of your derivation with the name of the inference rule. If its truth is unprovable, state so.

**Question 1. [5 pts]** $(A \otimes \mathbf{1}) \multimap A$

$$\dfrac{\dfrac{\dfrac{}{A \otimes \mathbf{1} \Vdash A \otimes \mathbf{1}}\,\mathsf{Hyp} \qquad \dfrac{\dfrac{}{\mathbf{1} \Vdash \mathbf{1}}\,\mathsf{Hyp} \quad \dfrac{}{A \Vdash A}\,\mathsf{Hyp}}{A, \mathbf{1} \Vdash A}\,\mathbf{1}\mathsf{E}}{A \otimes \mathbf{1} \Vdash A}\,\otimes\mathsf{E}}{\cdot \Vdash (A \otimes \mathbf{1}) \multimap A}\,\multimap\mathsf{I}$$

**Question 2. [5 pts]** $(A \oplus \mathbf{0}) \multimap A$

$$\dfrac{\dfrac{\dfrac{}{A \oplus \mathbf{0} \Vdash A \oplus \mathbf{0}}\,\mathsf{Hyp} \qquad \dfrac{}{A \Vdash A}\,\mathsf{Hyp} \qquad \dfrac{\dfrac{}{\mathbf{0} \Vdash \mathbf{0}}\,\mathsf{Hyp}}{\mathbf{0} \Vdash A}\,\mathbf{0}\mathsf{E}}{A \oplus \mathbf{0} \Vdash A}\,\oplus\mathsf{E}}{\cdot \Vdash (A \oplus \mathbf{0}) \multimap A}\,\multimap\mathsf{I}$$

**Question 3. [5 pts]** $(A \,\&\, A) \multimap A$

$$\dfrac{\dfrac{\dfrac{}{A \,\&\, A \Vdash A \,\&\, A}\,\mathsf{Hyp}}{A \,\&\, A \Vdash A}\,\&\mathsf{E}_L}{\cdot \Vdash A \,\&\, A \multimap A}\,\multimap\mathsf{I}$$

**Question 4. [5 pts]** $(A \oplus A) \multimap A$

$$\dfrac{\dfrac{\dfrac{}{A \oplus A \Vdash A \oplus A}\,\mathsf{Hyp} \qquad \dfrac{}{A \Vdash A}\,\mathsf{Hyp} \qquad \dfrac{}{A \Vdash A}\,\mathsf{Hyp}}{A \oplus A \Vdash A}\,\oplus\mathsf{E}}{\cdot \Vdash A \oplus A \multimap A}\,\multimap\mathsf{I}$$

**Question 5. [5 pts]** $(A \oplus (B \otimes C)) \multimap ((A \oplus B) \otimes (A \oplus C))$ *true* is provable. True or false?
False.

**Question 6. [5 pts]** $(A \oplus (B \,\&\, C)) \multimap ((A \oplus B) \,\&\, (A \oplus C))$ *true* is provable. True or false?
True.

**Question 7. [5 pts]** $(!A \otimes !B) \multimap (A \,\&\, B)$ *true* is provable. True or false?
True.

**Question 8. [5 pts]** $((A \multimap !A) \,\&\, (B \multimap !B)) \multimap ((A \otimes B) \multimap !(A \otimes B))$ *true* is provable. True or false?
False.

## Work sheet

## Work sheet

## Work sheet