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# CSE-433 Logic in Computer Science 2007
# Midterm exam — Sample Solution

- This is a closed-book exam. No other material is permitted.
- It consists of 4 problems worth a total of 140 points and an extra-credit problem worth 35 points.
- There are 16 pages in this exam, including 2 work sheets.
- Try to use work sheets before writing your answers. Write your answers clearly and legibly.
- You have 3 hours for this exam.

| | Problem 1 | Problem 2 | Problem 3 | Problem 4 | EC Problem | Total |
|---|---|---|---|---|---|---|
| Score | | | | | | |
| Max | 50 | 35 | 25 | 30 | 35 | 175 |

## 1 Short answers [50 pts]

Below we assume constructive logic, not classical logic, unless otherwise noted.

**Question 1. [5 pts]** $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ holds. True or false?
True.

**Question 2. [5 pts]** Give a proof of $((A \vee B) \supset C) \supset ((A \supset C) \wedge (B \supset C))\ \mathit{true}$ in the natural deduction style.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \vee B) \supset C\ \mathit{true}}\,x \quad \dfrac{\dfrac{}{A\ \mathit{true}}\,y}{A \vee B\ \mathit{true}}\,{\vee}\mathsf{I_L}}{C\ \mathit{true}}\,{\supset}\mathsf{E}}{A \supset C\ \mathit{true}}\,{\supset}\mathsf{I}^y \qquad \dfrac{\dfrac{\dfrac{}{(A \vee B) \supset C\ \mathit{true}}\,x \quad \dfrac{\dfrac{}{B\ \mathit{true}}\,z}{A \vee B\ \mathit{true}}\,{\vee}\mathsf{I_R}}{C\ \mathit{true}}\,{\supset}\mathsf{E}}{B \supset C\ \mathit{true}}\,{\supset}\mathsf{I}^z}{(A \supset C) \wedge (B \supset C)\ \mathit{true}}\,{\wedge}\mathsf{I}}{((A \vee B) \supset C) \supset ((A \supset C) \wedge (B \supset C))\ \mathit{true}}\,{\supset}\mathsf{I}^x$$

**Question 3. [5 pts]** Show a proof term reflecting your proof of $((A \vee B) \supset C) \supset ((A \supset C) \wedge (B \supset C))\ \mathit{true}$.

$$\lambda x\!:\!(A \vee B) \supset C.\, (\lambda y\!:\!A.\, x\ \mathsf{inl}_B\ y, \lambda z\!:\!B.\, x\ \mathsf{inr}_A\ z)$$

**Question 4. [5 pts]** Show a proof term of type $((A \supset C) \wedge (B \supset C)) \supset ((A \vee B) \supset C)$.

$$\lambda x\!:\!(A \supset C) \wedge (B \supset C).\, \lambda y\!:\!A \vee B.\, \mathsf{case}\ y\ \mathsf{of}\ \mathsf{inl}\ z_1.\, \mathsf{fst}\ x\ z_1 \mid \mathsf{inr}\ z_2.\, \mathsf{snd}\ x\ z_2$$

**Question 5. [5 pts]** What is the result of applying a commuting conversion to the following derivation tree where the rule $R$ is assumed to be an elimination rule?

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D} \\ \exists x. A\ \mathit{true}\end{matrix} \qquad \begin{matrix}\dfrac{}{[a/x]A\ \mathit{true}}\,w \\ \vdots \\ C\ \mathit{true}\end{matrix}}{C\ \mathit{true}}\,\exists\mathsf{E}^{a,w}}{C'\ \mathit{true}}\,R \qquad \Longrightarrow_C$$

$$\dfrac{\begin{matrix}\mathcal{D} \\ \exists x. A\ \mathit{true}\end{matrix} \qquad \begin{matrix}\dfrac{}{[a/x]A\ \mathit{true}}\,w \\ \vdots \\ \dfrac{C\ \mathit{true}}{C'\ \mathit{true}}\,R\end{matrix}}{C'\ \mathit{true}}\,\exists\mathsf{E}^{a,w}$$

**Question 6. [5 pts]** Specify a commuting conversion for $\bot$.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D} \\ \bot\ \mathit{true}\end{matrix}}{C\ \mathit{true}}\,\bot\mathsf{E}}{C'\ \mathit{true}}\,R \qquad \Longrightarrow_C \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \bot\ \mathit{true}\end{matrix}}{C'\ \mathit{true}}\,\bot\mathsf{E}$$

**Question 7. [5 pts]** Apply $\beta$-reductions and commuting conversions to reduce the following proof term to the simplest form. We assume that the proof term is well-typed.

$$(\mathsf{case}\ x\ \mathsf{of}\ \mathsf{inl}\ y_1.\, \lambda z\!:\!A \supset A.\, z\ y_1 \mid \mathsf{inr}\ y_2.\, y_2)\ \lambda w\!:\!A.\, w$$

$$(\mathsf{case}\ x\ \mathsf{of}\ \mathsf{inl}\ y_1.\, y_1 \mid \mathsf{inr}\ y_2.\, y_2\ (\lambda w\!:\!A.\, w))$$

**Question 8. [5 pts]** There is only one normal proof of $(A \supset B) \supset (A \supset B)$ *true*. True or false?
False.

**Question 9. [5 pts]** Convert a proof term $\lambda x\!:\!A \wedge B.\, x$ into long normal form.

$$\lambda x\!:\!A \wedge B.\, (\mathsf{fst}\ x, \mathsf{snd}\ x)$$

**Question 10. [5 pts]** If $A$ *true* is not provable in classical logic, $A$ *true* is not provable in constructive logic, either. True or false?
True.

## 2 Another elimination rule for conjunction [35 pts]

In class, we defined two elimination rules for conjunction $\wedge$:

$$\frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}}\ \wedge\mathsf{E_L} \qquad \frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}}\ \wedge\mathsf{E_R}$$

It turns out that we can combine the two rules into one. In this problem, we will devise such an elimination rule and show the local soundness and completeness properties for that rule. We will also design a new proof term for the new elimination rule and derive a $\beta$-reduction and an $\eta$-expansion.

For the introduction rule, we use the rule $\wedge\mathsf{I}$ that we learned in class:

$$\frac{A\ \mathit{true} \quad B\ \mathit{true}}{A \wedge B\ \mathit{true}}\ \wedge\mathsf{I}$$

Write your answer in the natural deduction style, but do not use hypothetical judgments. Be careful not to destroy the orthogonality of the system, *i.e.*, do not use any other connectives in your elimination rule.

**Question 1. [10 pts]** Propose an elimination rule for conjunction $\wedge$ that combines the two rules $\wedge\mathsf{E_L}$ and $\wedge\mathsf{E_R}$. Hint: When applying the rule to $A \wedge B\ \mathit{true}$, the conclusion of the rule can be another judgment $C\ \mathit{true}$, as is the case in the elimination rule $\vee\mathsf{E}$ for disjunction $\vee$.

$$\frac{A \wedge B\ \mathit{true} \qquad \begin{array}{c} \overline{A\ \mathit{true}}^{\,x} \quad \overline{B\ \mathit{true}}^{\,y} \\ \vdots \\ C\ \mathit{true} \end{array}}{C\ \mathit{true}}\ \wedge\mathsf{E}^{x,y}$$

**Question 2. [5 pts]** Show a local reduction $\Longrightarrow_R$ for your elimination rule.

$$\frac{\dfrac{\begin{array}{cc} \mathcal{D} & \mathcal{E} \\ A\ \mathit{true} & B\ \mathit{true} \end{array}}{A \wedge B\ \mathit{true}}\ \wedge\mathsf{I} \qquad \begin{array}{c} \overline{A\ \mathit{true}}^{\,x} \quad \overline{B\ \mathit{true}}^{\,y} \\ \vdots \\ C\ \mathit{true} \end{array}}{C\ \mathit{true}}\ \wedge\mathsf{E}^{x,y} \quad \Longrightarrow_R \quad \begin{array}{c} \begin{array}{cc} \mathcal{D} & \mathcal{E} \\ A\ \mathit{true} & B\ \mathit{true} \end{array} \\ \vdots \\ C\ \mathit{true} \end{array}$$

**Question 3. [5 pts]** Show a local expansion $\Longrightarrow_E$ for your elimination rule.

$$\begin{array}{c} \mathcal{D} \\ A \wedge B\ \mathit{true} \end{array} \quad \Longrightarrow_E \quad \frac{\begin{array}{c} \mathcal{D} \\ A \wedge B\ \mathit{true} \end{array} \qquad \dfrac{\overline{A\ \mathit{true}}^{\,x} \quad \overline{B\ \mathit{true}}^{\,y}}{A \wedge B\ \mathit{true}}\ \wedge\mathsf{I}}{A \wedge B\ \mathit{true}}\ \wedge\mathsf{E}^{x,y}$$

**Question 4. [5 pts]** Design a proof term for your elimination rule. That is, rewrite your elimination rule into a typing rule, introducing a new form of proof term as necessary.

$$\frac{M : A \wedge B \qquad \begin{array}{c} \overline{x : A} \quad \overline{y : B} \\ \vdots \\ N : C \end{array}}{\mathsf{let}\ (x, y) = M\ \mathsf{in}\ N : C}\ \wedge\mathsf{E}$$

**Question 5. [5 pts]** Show a $\beta$-reduction $\Longrightarrow_\beta$ for your elimination rule.

$\mathsf{let}\ (x, y) = (M_1, M_2)\ \mathsf{in}\ N \quad \Longrightarrow_\beta \quad [M_1/x][M_2/y]N$

**Question 6. [5 pts]** Show an $\eta$-expansion $\Longrightarrow_\eta$ for your elimination rule.

$M : A \wedge B \quad \Longrightarrow_\eta \quad \mathsf{let}\ (x, y) = M\ \mathsf{in}\ (x, y)$

# 3 Eliminating nested quantifiers in first-order logic [25 points]

**Question 1. [5 pts]** Let us assume a binary predicate $\mathit{Love}$ and a unary predicate $\mathit{Happy}$. $\mathit{Love}(t, t')$ means that $t$ loves $t'$, and $\mathit{Happy}(t)$ means that $t$ is happy.

Translate the following sentence into a first-order proposition:

*For every (person) $y$, if there exists another (person) $x$ who loves $y$, then $y$ is happy.*

$\forall y.(\exists x.\mathit{Love}(x, y)) \supset \mathit{Happy}(y)$

**Question 2. [5 pts]** Suppose that term variable $x$ is not free in proposition $A$, but free in proposition $B$. That is, we have $[t/x]A = A$, but $[t/x]B \neq B$ in general for an arbitrary term $t$. Recall that quantifiers $\forall$ and $\exists$ have the lowest operator precedence.

Give a proof of $(\forall x.B \supset A) \supset (\exists x.B) \supset A\ \mathit{true}$ in the natural deduction style.

$$\dfrac{\dfrac{\dfrac{\overline{\exists x.B}^{\,v} \quad \dfrac{\dfrac{\overline{\forall x.B \supset A}^{\,w}}{[a/x]B \supset A\ \mathit{true}}\ \forall\mathsf{E} \quad \overline{[a/x]B}^{\,z}}{A\ \mathit{true}}\ \supset\mathsf{E}}{A\ \mathit{true}}\ \exists\mathsf{E}^{a,z}}{(\exists x.B) \supset A\ \mathit{true}}\ \supset\mathsf{I}^{v}}{(\forall x.B \supset A) \supset (\exists x.B) \supset A\ \mathit{true}}\ \supset\mathsf{I}^{w}$$

**Question 3. [5 pts]** Under the same assumption on $A$ and $B$ as in the previous question, give a proof term of type $((\exists x.B) \supset A) \supset (\forall x.B \supset A)$.

$$\lambda w : (\exists x.B) \supset A.\, \lambda x.\, \lambda y : B.\, w\ \langle x, y \rangle$$

**Question 4. [5 pts]** Questions 2 and 3 show that we have the following logical equivalence:

$$(\exists x.B) \supset A \equiv \forall x.B \supset A$$

Use this logical equivalence to rewrite your answer to Question 1 into another (logically equivalent) first-order proposition.

$\forall y.\forall x.\mathit{Love}(x, y) \supset \mathit{Happy}(y)$

**Question 5. [5 pts]** Translate your answer to Question 4 back into English!

*For every (person) $y$ and $x$, if $x$ loves $y$, then $y$ is happy.*

# 4 Harrop's rule [30 pts]

In constructive logic, there are some admissible rules that are not derivable. An example of such a rule is

$$\frac{\neg A \supset (B \vee C)\ \mathit{true}}{(\neg A \supset B) \vee (\neg A \supset C)\ \mathit{true}}\ \mathit{Harrop}$$

where $A, B$, and $C$ are all propositions. This rule, which we call Harrop's rule, is not derivable because it is impossible to prove $(\neg A \supset (B \vee C)) \supset (\neg A \supset B) \vee (\neg A \supset C)\ \mathit{true}$.

In this problem, we will show that Harrop's rule is not derivable. (We plan to prove in the final exam that Harrop's rule is indeed admissible!) You will need to use the following inference rules for neutral and normal judgments.

$$\frac{\begin{array}{c}\overline{A\downarrow}^{\,x}\\ \vdots \\ B\uparrow\end{array}}{A \supset B\uparrow}\supset\mathsf{I}^x_\uparrow \qquad \frac{A \supset B\downarrow \quad A\uparrow}{B\downarrow}\supset\mathsf{E}_\downarrow \qquad \frac{A\uparrow \quad B\uparrow}{A \wedge B\uparrow}\wedge\mathsf{I}_\uparrow \qquad \frac{A \wedge B\downarrow}{A\downarrow}\wedge\mathsf{E}_{\mathsf{L}\downarrow} \qquad \frac{A \wedge B\downarrow}{B\downarrow}\wedge\mathsf{E}_{\mathsf{R}\downarrow}$$

$$\frac{A\uparrow}{A \vee B\uparrow}\vee\mathsf{I}_{\mathsf{L}\uparrow} \qquad \frac{B\uparrow}{A \vee B\uparrow}\vee\mathsf{I}_{\mathsf{R}\uparrow} \qquad \frac{A \vee B\downarrow \quad \begin{array}{c}\overline{A\downarrow}^{\,x}\\ \vdots \\ C\uparrow\end{array} \quad \begin{array}{c}\overline{B\downarrow}^{\,y}\\ \vdots \\ C\uparrow\end{array}}{C\uparrow}\vee\mathsf{E}^{x,y}_\uparrow$$

$$\frac{}{\top\uparrow}\top\mathsf{I}_\uparrow \qquad \frac{\bot\downarrow}{C\uparrow}\bot\mathsf{E}_\uparrow \qquad \frac{A\downarrow}{A\uparrow}\downarrow\uparrow \qquad \frac{\begin{array}{c}\overline{A\downarrow}^{\,x}\\ \vdots \\ \bot\uparrow\end{array}}{\neg A\uparrow}\neg\mathsf{I}_\uparrow^{\,x} \qquad \frac{\neg A\downarrow \quad A\uparrow}{\bot\downarrow}\neg\mathsf{E}_\downarrow$$

- You should annotate every part of your derivation with the name of the inference rule (*e.g.*, $\supset\mathsf{E}_\downarrow$) and also a label if applicable (*e.g.*, $\supset\mathsf{I}^x_\uparrow$). In particular, you should annotate each hypothesis with some variable (*e.g.*, $\overline{A\downarrow}^{\,x}$).
- In your argument, you may use the fact that it is impossible to derive $\neg A\uparrow$ from $\neg A \supset B \vee C\downarrow$ alone.
- In your argument, you may use the fact that it is impossible to derive $B\uparrow$ or $C\uparrow$ from $B \vee C\downarrow$ alone.

Recall that in constructive logic, $A\ \mathit{true}$ is provable if and only if $A\uparrow$ is provable. Therefore we will show that Harrop's rule is not derivable by showing that there is no proof of

$$(\neg A \supset (B \vee C)) \supset (\neg A \supset B) \vee (\neg A \supset C)\uparrow.$$

*Proof.* In order to prove $(\neg A \supset (B \vee C)) \supset (\neg A \supset B) \vee (\neg A \supset C)\uparrow$, we have to derive $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$ from $\neg A \supset (B \vee C)\downarrow$:

$$\frac{\mathcal{D}\left\{\begin{array}{c}\overline{\neg A \supset (B \vee C)\downarrow}^{\,x}\\ \vdots \\ (\neg A \supset B) \vee (\neg A \supset C)\uparrow\end{array}\right.}{(\neg A \supset (B \vee C)) \supset (\neg A \supset B) \vee (\neg A \supset C)\uparrow}\supset\mathsf{I}^x_\uparrow$$

We need to show that it is impossible to fill $\vdots$ by exploring *all* possibilities for applying introduction or elimination rules in $\mathcal{D}$.

*Case* 1. Starting from $\mathcal{D}$, we can apply the rule ${\supset}\mathsf{E}_{\downarrow}$ to $\overline{\neg A \supset (B \vee C)\downarrow}^{\,x}$. Show the resultant derivation after applying the rule ${\supset}\mathsf{E}_{\downarrow}$ to $\overline{\neg A \supset (B \vee C)\downarrow}^{\,x}$; you should duplicate $\overline{\neg A \supset (B \vee C)\downarrow}^{\,x}$ somewhere (5 pts):

$$
\dfrac{\begin{array}{c}
\dfrac{\overline{\neg A \supset (B \vee C)\downarrow}^{\,x} \qquad \begin{array}{c}\overline{\neg A \supset (B \vee C)\downarrow}^{\,x} \\ \vdots \\ \neg A\uparrow\end{array}}{B \vee C\downarrow}\ {\supset}\mathsf{E}_{\downarrow} \\ \vdots \\ (\neg A \supset B) \vee (\neg A \supset C)\uparrow
\end{array}}{(\neg A \supset (B \vee C)) \supset (\neg A \supset B) \vee (\neg A \supset C)\uparrow}\ {\supset}\mathsf{I}^{x}_{\uparrow}
$$

We may be able to fill in the gap between $B \vee C\downarrow$ and $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$. However, we cannot complete the proof because it is impossible to derive $\neg A\uparrow$ from $\neg A \supset (B \vee C)\downarrow$ alone.

*Case* 2. Starting from $\mathcal{D}$ again, we can use the rule $\vee\mathsf{I}_{\mathsf{L}\uparrow}$ to derive $(\neg A \supset B) \vee (\neg A \supset C)\uparrow$. Show the resultant derivation (5 pts):

$$
\dfrac{\dfrac{\begin{array}{c}\overline{\neg A \supset (B \vee C)\downarrow}^{\,x} \\ \vdots \\ \neg A \supset B\uparrow\end{array}}{(\neg A \supset B) \vee (\neg A \supset C)\uparrow}\ \vee\mathsf{I}_{\mathsf{L}\uparrow}}{(\neg A \supset (B \vee C)) \supset (\neg A \supset B) \vee (\neg A \supset C)\uparrow}\ {\supset}\mathsf{I}^{x}_{\uparrow}
$$

If we apply the rule ${\supset}\mathsf{E}_{\downarrow}$ to $\overline{\neg A \supset (B \vee C)\downarrow}^{\,x}$, we get stuck as shown in *Case* 1. Therefore the only way to proceed is by applying the rule ${\supset}\mathsf{I}_{\uparrow}$ to $\neg A \supset B$ (fill in the blank). Show the resultant derivation (5pts):

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\overline{\neg A \supset (B \vee C)\downarrow}^{\,x} \quad \overline{\neg A\downarrow}^{\,y} \\ \vdots \\ B\uparrow\end{array}}{\neg A \supset B\uparrow}\ {\supset}\mathsf{I}^{y}_{\uparrow}}{(\neg A \supset B) \vee (\neg A \supset C)\uparrow}\ \vee\mathsf{I}_{\mathsf{L}\uparrow}}{(\neg A \supset (B \vee C)) \supset (\neg A \supset B) \vee (\neg A \supset C)\uparrow}\ {\supset}\mathsf{I}^{x}_{\uparrow}
$$

Now the only way to proceed is by applying the rule ${\supset}\mathsf{E}_{\downarrow}$ to $\overline{\neg A \supset (B \vee C)\downarrow}^{\,x}$ (fill in the blank). Show the resultant derivation: (5 pts)

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\overline{\neg A \supset (B \vee C)\downarrow}^{\,x} \quad \overline{\neg A\downarrow\uparrow}^{\,y}}{B \vee C\downarrow}\ {\supset}\mathsf{E}_{\downarrow} \\ \vdots \\ B\uparrow\end{array}}{\neg A \supset B\uparrow}\ {\supset}\mathsf{I}^{y}_{\uparrow}}{(\neg A \supset B) \vee (\neg A \supset C)\uparrow}\ \vee\mathsf{I}_{\mathsf{L}\uparrow}}{(\neg A \supset (B \vee C)) \supset (\neg A \supset B) \vee (\neg A \supset C)\uparrow}\ {\supset}\mathsf{I}^{x}_{\uparrow}
$$

Then we cannot complete the proof because it is impossible to derive $B\!\uparrow$ from $B \vee C\!\downarrow$ alone. (fill in the blank).

*Case* 3. Starting from $\mathcal{D}$ again, we can use the rule $\vee I_{R\uparrow}$ to derive $(\neg A \supset B) \vee (\neg A \supset C)\!\uparrow$. The remaining steps are similar to those in *Case* 2. Show the final derivation at which we get stuck: (5 pts)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg A \supset (B \vee C)\!\downarrow}^{\,x} \quad \overline{\neg A\!\downarrow\uparrow}^{\,y}}{B \vee C\!\downarrow}\supset E_{\downarrow}}{\vdots \atop C\!\uparrow}}{\neg A \supset C\!\uparrow}\supset I_{\uparrow}^{y}}{(\neg A \supset B) \vee (\neg A \supset C)\!\uparrow}\vee I_{R\uparrow}}{(\neg A \supset (B \vee C)) \supset (\neg A \supset B) \vee (\neg A \supset C)\!\uparrow}\supset I_{\uparrow}^{x}
$$

We cannot complete the proof because it is impossible to derive $C\!\uparrow$ from $B \vee C\!\downarrow$ alone. (fill in the blank) (3 pts).

The above three cases are the only possibilities for applying introduction and elimination rules in $\mathcal{D}$. Therefore there is no proof of $(\neg A \supset (B \vee C)) \supset (\neg A \supset B) \vee (\neg A \supset C)\!\uparrow$, which means that Harrop's rule is not derivable.

# 5 Sequent calculus [Extra-credit 35 points]

This question tests your understanding of normal proofs and neutral proofs. We will investigate an alternative way, called *sequent calculus*, of representing deductions of normal proofs. It plays a central role in the study of meta-theoretic properties of constructive logic.

## Introduction

Recall that we write $A \uparrow$ for a normal proof of $A\ true$ and $A \downarrow$ for a neutral proof of $A\ true$. At any point during the construction of a normal proof, there are three kinds of judgments available:

1. hypotheses which are all neutral proofs: $A_1 \downarrow, \cdots, A_m \downarrow$.
2. neutral proofs derived from hypotheses by elimination rules: $B_1 \downarrow, \cdots, B_n \downarrow$.
3. a normal proof yet to be built: $C \uparrow$.

Pictorially we would like to fill the gap between $B_1 \downarrow, \cdots, B_n \downarrow$ and $C \uparrow$:

$$\begin{array}{c} \overline{A_1 \downarrow} \qquad \cdots \qquad \overline{A_m \downarrow} \\ \text{(by elimination rules)} \\ \overline{B_1 \downarrow} \quad \cdots \quad \overline{B_n \downarrow} \\ \vdots \\ \text{(yet to be filled)} \\ \vdots \\ C \uparrow \end{array}$$

We represent the above partial deduction as a new judgment, called a *sequent*,

$$A_1, \cdots, A_m, B_1, \cdots, B_n \longrightarrow C.$$

Conceptually it means "we are building a normal proof of $C$ from a collection of hypotheses $A_1, \cdots, A_m$ and neutral proofs of $B_1, \cdots, B_n$ derived from hypotheses by elimination rules." Since hypotheses are also neutral proofs, we cease to distinguish between hypotheses and neutral proofs derived from them in the left-hand side. Thus

$$A_1, \cdots, A_n \longrightarrow C$$

means "we are building a normal proof of $C$ from a collection of neutral proofs of $A_1, \cdots, A_n$," where $A_i$ is either a hypothesis or a neutral proof derived from hypotheses by elimination rules. As usual, the order of propositions in the left-hand side does not matter. The goal of this question is to develop the inference rules for the new form of judgment $A_1, \cdots, A_n \longrightarrow C$.

## Example 1: Elimination rule for the left-hand side

Consider $A_1, \cdots, A_n, A \wedge B \longrightarrow C$, which represents the following partial deduction:

$$\begin{array}{c} \overline{A_1 \downarrow} \qquad \cdots \qquad \overline{A_n \downarrow} \quad \overline{A \wedge B \downarrow} \\ \vdots \\ \text{(yet to be filled)} \\ \vdots \\ C \uparrow \end{array}$$

Suppose that we decide to apply the left $\wedge$-elimination rule to $A \wedge B \downarrow$. It gives us the following partial deduction:

$$\overline{A_1 \downarrow} \qquad \cdots \qquad \overline{A_n \downarrow} \quad \dfrac{\overline{A \wedge B \downarrow}}{A \downarrow} \wedge\mathsf{E_L}$$

$$\begin{array}{c} \vdots \\ \text{(yet to be filled)} \\ \vdots \\ C \uparrow \end{array}$$

Now $A \downarrow$ is another neutral proof available, so the new partial deduction is represented as $A_1, \cdots, A_n, A \wedge B, A \longrightarrow C$.

What we have seen in this example is that the problem of proving $A_1, \cdots, A_n, A \wedge B \longrightarrow C$ is reduced to the problem of proving $A_1, \cdots, A_n, A \wedge B, A \longrightarrow C$. Thus we propose the following inference rule $\wedge L_L$ for the new form of judgment:

$$\frac{A_1, \cdots, A_n, A \wedge B, A \longrightarrow C}{A_1, \cdots, A_n, A \wedge B \longrightarrow C} \wedge L_L$$

It is best to read the inference rule *in the bottom-up way*, not in the top-down way.

## Example 2: Introduction rule for the right-hand side

Consider $A_1, \cdots, A_n \longrightarrow A \vee B$, which represents the following partial deduction:

$$\begin{array}{c} \overline{A_1 \downarrow} \qquad \cdots \qquad \overline{A_n \downarrow} \\ \vdots \\ \text{(yet to be filled)} \\ \vdots \\ A \vee B \uparrow \end{array}$$

Suppose that we decide to apply the left $\vee$-introduction rule to $A \vee B \uparrow$. It gives us the following partial deduction:

$$\begin{array}{c} \overline{A_1 \downarrow} \qquad \cdots \qquad \overline{A_n \downarrow} \\ \vdots \\ \text{(yet to be filled)} \\ \vdots \\ \dfrac{A \uparrow}{A \vee B \uparrow} \vee\mathsf{I_L} \end{array}$$

Now $A \uparrow$ is a new goal, so the problem of proving $A_1, \cdots, A_n \longrightarrow A \vee B$ is reduced to the problem of proving $A_1, \cdots, A_n \longrightarrow A$. Thus we propose the following inference rule $\vee R_L$ for the new form of judgment:

$$\frac{A_1, \cdots, A_n \longrightarrow A}{A_1, \cdots, A_n \longrightarrow A \vee B} \vee R_L$$

As before, it is best to read the inference rule in the bottom-up way, not in the top-down way.

## Example 3: From a neutral proof to a normal proof

Consider $A_1, \cdots, A_n \longrightarrow A_n$ (where the right-hand side $A_n$ matches one of the propositions in the left-hand side). It represents the following partial deduction:

$$\begin{array}{ccc} \overline{A_1 \downarrow} & \cdots & \overline{A_n \downarrow} \\ & \vdots & \\ & \text{(yet to be filled)} & \\ & \vdots & \\ & A_n \uparrow & \end{array}$$

Now we can immediately bridge the gap, since we can convert $A_n \downarrow$ to $A_n \uparrow$:

$$\overline{A_1 \downarrow} \qquad \cdots \qquad \dfrac{\overline{A_n \downarrow}}{A_n \uparrow} \downarrow\uparrow \qquad \text{(proof completed)}$$

Hence the proof of $A_1, \cdots, A_n \longrightarrow A_n$ requires no further evidence, and we propose the following inference rule *Init* for the new form of judgment:

$$\dfrac{}{A_1, \cdots, A_n \longrightarrow A_n} \; \mathit{Init}$$

Note that this rule *Init* does _not_ correspond to the use of hypotheses in natural deductions.

**Question 1. [10 pts]** Using the above three rules $\wedge L_L$, $\vee R_L$, and *Init*, prove $A \wedge B \longrightarrow A \vee B$. Begin with $A \wedge B \longrightarrow A \vee B$ and proceed in the bottom-up way. Your proof should end with an application of the rule *Init*:

$$\dfrac{}{A \wedge B \longrightarrow A \vee B}$$

*Answer.*

$$\dfrac{\dfrac{\dfrac{}{A \wedge B, A \longrightarrow A} \; \mathit{Init}}{A \wedge B, A \longrightarrow A \vee B} \; \vee R_L}{A \wedge B \longrightarrow A \vee B} \; \wedge L_L \qquad \text{or} \qquad \dfrac{\dfrac{\dfrac{}{A \wedge B, A \longrightarrow A} \; \mathit{Init}}{A \wedge B \longrightarrow A} \; \wedge L_L}{A \wedge B \longrightarrow A \vee B} \; \vee R_L$$

□

From now on, we write $\Gamma$ for a collection of propositions $A_1, \cdots, A_n$.

**Question 2. [5 pts]** The inference rules corresponding to the $\wedge$-elimination rules are as follows:

$$\dfrac{\Gamma, A \wedge B, A \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \; \wedge L_L \qquad \dfrac{\Gamma, A \wedge B, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C} \; \wedge L_R$$

Propose an inference rule corresponding to the $\wedge$-introduction rule:

$$\frac{}{\Gamma \longrightarrow A \wedge B} \wedge R$$

*Answer.*

$$\frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \wedge R$$

□

**Question 3. [10 pts]** The inference rule corresponding to the $\supset$-introduction rule is as follows:

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B} \supset R$$

Propose an inference rule corresponding to the $\supset$-elimination rule:

$$\frac{}{\Gamma, A \supset B \longrightarrow C} \supset L$$

*Answer.*

$$\frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C} \supset L$$

□

**Question 4. [5 pts]** The inference rules corresponding to the $\vee$-introduction rules are as follows:

$$\frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B} \vee R_L \qquad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B} \vee R_R$$

Propose an inference rule corresponding to the $\vee$-elimination rule:

$$\frac{}{\Gamma, A \vee B \longrightarrow C} \vee L$$

*Answer.*

$$\frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee L \qquad \text{or} \qquad \frac{\Gamma, A \vee B, A \longrightarrow C \quad \Gamma, A \vee B, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C} \vee L$$

□

**Question 5. [5 pts]**

The two remaining inference rules correspond to the $\top$-introduction rule and the $\bot$-elimination rule:

$$\frac{}{\Gamma \longrightarrow \top}\ \top R \qquad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$

It is known that the above system of inference rules is sound and complete with respect to constructive logic: $\cdot \longrightarrow A$ is provable if and only if $A$ *true* is provable in constructive logic. As an immediate consequence, we can show that constructive logic itself is sound. Show that there is no derivation of $\cdot \longrightarrow \bot$ (and conclude that $\bot$ *true* is not provable in constructive logic):

$$\frac{???}{\cdot \longrightarrow \bot}$$

*Answer.* There is no rule to apply to $\cdot \longrightarrow \bot$. Therefore $\cdot \longrightarrow \bot$ is impossible to prove. □

## Work sheet

## Work sheet