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# CSE-433 Logic in Computer Science 2010
# Final exam

- This is a closed-book exam. No other material is permitted.
- It consists of 5 problems worth a total of 210 points.
- There are 14 pages in this exam, including 3 work sheets.
- Try to use work sheets before writing your answers. Write your answers clearly and legibly.
- You have 3 hours for this exam.

| | Problem 1 | Problem 2 | Problem 3 | Problem 4 | Problem 5 | Total |
|---|---|---|---|---|---|---|
| Score | | | | | | |
| Max | 30 | 30 | 50 | 50 | 50 | 210 |

# 1 Definitional equality and terms [30pts]

We write $t = s$ to mean that terms $t$ and $s$ are definitionally equal.

$$\frac{t \Longrightarrow_\beta^* r \quad s \Longrightarrow_\beta^* r}{t = s}\ \mathit{DefEq}\mathsf{I}$$

In the introduction rule $\mathit{DefEq}\mathsf{I}$, the judgment $t \Longrightarrow_\beta^* r$ means that $t$ reduces to $r$ by zero or more $\beta$-reductions. Here we assume that $\beta$-reductions may be applied to subterms of the term being reduced.

**Question 1. [5 pts]** Definitional equality is transitive. Yes or no? Explain why. (-5 pts for a wrong answer)

**Question 2. [5 pts]** Consider terms for datatype nat:

$$\text{term} \quad t \quad ::= \quad \cdots \mid \mathbf{0} \mid \mathbf{s}(t) \mid \mathbf{rec}\ f(t)\ \mathbf{of}\ f(\mathbf{0}) \Rightarrow t \mid f(\mathbf{s}(x)) \Rightarrow t$$

Write two $\beta$-reductions for datatype nat:

$$\Longrightarrow_\beta$$

$$\Longrightarrow_\beta$$

**Question 3. [5 pts]** Define a function $plus$ of datatype nat → nat → nat adding two natural numbers.

$plus\ =$

**Question 4. [5 pts]** $plus\ \mathbf{0}\ x = x$ holds. True or false? (-5 pts for a wrong answer)

**Question 5. [5 pts]** $plus\ x\ \mathbf{0} = x$ holds. True or false? (-5 pts for a wrong answer)

**Question 6. [5 pts]** $plus\ \mathbf{s}(x)\ y = \mathbf{s}(plus\ x\ y)$ holds. True or false? (-5 pts for a wrong answer)

## 2 Local and global soundness and completeness [30pts]

**Question 1. [5 pts]** Explain local soundness in no more than 100 words.

**Question 2. [10 pts]** Explain global soundness in no more than 100 words.

**Question 3. [5 pts]** Explain local completeness in no more than 100 words.

**Question 4. [10 pts]** Explain global completeness in no more than 100 words.

## 3 First-order logic [50 pts]

Consider the natural deduction system for first-order logic (without datatypes). We use $t$ for terms, $x$ for term variables, and $a$ for parameters.

$$\frac{[a/x]A\ true}{\forall x.A\ true}\ \forall\mathsf{I}^a \qquad \frac{\forall x.A\ true}{[t/x]A\ true}\ \forall\mathsf{E} \qquad \frac{[t/x]A\ true}{\exists x.A\ true}\ \exists\mathsf{I} \qquad \frac{\exists x.A\ true \qquad \begin{array}{c}\overline{[a/x]A\ true}^{\,w}\\ \vdots \\ C\ true\end{array}}{C\ true}\ \exists\mathsf{E}^{a,w}$$

**Question 1. [10 pts]**

Propose proof terms and their typing rules for first-order logic. Do not use hypothetical judgments with contexts.

**Question 2. [10 pts]**

Propose four inference rules for normal and neutral judgments ($A\uparrow$ and $A\downarrow$) in the natural deduction style. Do not use hypothetical judgments with contexts.

**Question 3. [20 pts]**

Propose four inference rules for sequents $\Gamma \longrightarrow C$ in the sequent calculus.

**Question 4. [10 pts]**

Suppose that term variable $x$ is found in proposition $A$ and not in proposition $B$. That is, we have $[t/x]A \neq A$ and $[t/x]B = B$ in general for an arbitrary term $t$. Also recall that quantifiers have the lowest operator precedence, so, for example, we have $\forall x.A \supset B = \forall x.(A \supset B)$.

Give a proof of $(\exists x.A) \supset B \longrightarrow \forall x.A \supset B$ using the rules that you have proposed. If not provable, state so.

# 4 Classical logic [50 pts]

In this problem, we use contexts $\Gamma$ and $\Delta$ defined as follows:

$$\begin{array}{rcl} \Gamma & ::= & \cdot \mid \Gamma, x : A \\ \Delta & ::= & \cdot \mid \Delta, x : A \; \mathit{false} \end{array}$$

We use $\Gamma; \Delta \vdash_{\mathsf{K}} C \; \mathit{true}$ for typechecking proof terms in classical logic and $\Gamma \vdash_{\mathsf{I}} M : C$ for typechecking proof terms in constructive logic.

Here are the rules $\mathsf{Contra}\!\uparrow$ and $\mathsf{Contra}\!\downarrow$ given in the Course Notes:

$$\frac{\Gamma; \Delta, A \; \mathit{false} \vdash_{\mathsf{K}} A \; \mathit{true}}{\Gamma; \Delta \vdash_{\mathsf{K}} A \; \mathit{true}} \; \mathsf{Contra}\!\uparrow \qquad \frac{\Gamma; \Delta, A \; \mathit{false} \vdash_{\mathsf{K}} A \; \mathit{true}}{\Gamma; \Delta, A \; \mathit{false} \vdash_{\mathsf{K}} C \; \mathit{true}} \; \mathsf{Contra}\!\downarrow$$

**Question 1. [10 pts]** Use the rules $\mathsf{Contra}\!\uparrow$ and $\mathsf{Contra}\!\downarrow$ to show that the rule $\mathsf{DNE}$ is derivable.

$$\frac{}{\cdot; \cdot \vdash_{\mathsf{K}} \neg\neg A \supset A \; \mathit{true}}$$

**Question 2. [20 pts]** We use the following double-negation translation for the fragment of propositional logic with implication:

$$\begin{array}{rcl} P^\circ & = & P \\ (A \supset B)^\circ & = & A^\circ \supset \neg\neg B^\circ \end{array}$$

For each case below, complete the CPS translation $M^\circ$ of a given proof term $M$. Your translation should satisfy the invariant specified in the following theorem:

**Theorem.** *If $\Gamma; \Delta \vdash_{\mathsf{K}} M : C$, there exists a proof term $M^\circ$ such that $\Gamma^\circ, \neg\Delta^\circ \vdash_{\mathsf{I}} M^\circ : \neg\neg C^\circ$ where*

$$\begin{array}{rcl} \Gamma^\circ & = & \{x : A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ & = & \{x : \neg A^\circ \mid x : A\ \textit{false} \in \Delta\}. \end{array}$$

Case $\dfrac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathsf{K}} x : A}$ Hyp

$x^\circ =$

Case $\dfrac{\Gamma, x : A; \Delta \vdash_{\mathsf{K}} M : B}{\Gamma; \Delta \vdash_{\mathsf{K}} \lambda x{:}A.\, M : A \supset B}$ $\supset$I

$(\lambda x{:}A.\, M)^\circ =$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathsf{K}} N : A}{\Gamma; \Delta \vdash_{\mathsf{K}} M\ N : B}$ $\supset$E

$(M\ N)^\circ =$

Case $\dfrac{\Gamma; \Delta, x : A\ \textit{false} \vdash_{\mathsf{K}} M : A}{\Gamma; \Delta \vdash_{\mathsf{K}} \mathsf{callcc}\ x : A\ \textit{false}.\, M : A}$ Callcc

$(\mathsf{callcc}\ x : A\ \textit{false}.\, M)^\circ =$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \quad x : A\ \textit{false} \in \Delta}{\Gamma; \Delta \vdash_{\mathsf{K}} \mathsf{throw}\ M\ \mathsf{to}\ x : C}$ Throw

$(\mathsf{throw}\ M\ \mathsf{to}\ x)^\circ =$

**Question 3. [20 pts]** The above CPS transformation corresponds to the call-by-value reduction strategy. To be specific, the translation of $M\ N$ specifies that we finish evaluating $N$ before we apply the function from $M$ to the result of evaluating $N$.

In this question, we will develop a variant of the CPS translation that corresponds to the call-by-name reduction strategy. We use the following double-negation translation (known as *Kolmogorov double negation translation*) in which $(A \supset B)^\circ$ places $\neg\neg$ before both $A^\circ$ and $B^\circ$:

$$\begin{array}{rcl} P^\circ & = & P \\ (A \supset B)^\circ & = & \neg\neg A^\circ \supset \neg\neg B^\circ \end{array}$$

For each case below, complete the CPS translation $M^\circ$ of a given proof term $M$. Your translation should satisfy the invariant specified in the following theorem:

**Theorem.** *If $\Gamma; \Delta \vdash_{\mathsf{K}} M : C$, there exists a proof term $M^\circ$ such that $\Gamma^\circ, \neg\Delta^\circ \vdash_{\mathsf{I}} M^\circ : \neg\neg C^\circ$ where*

$$\begin{array}{rcl} \Gamma^\circ & = & \{x : \neg\neg A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ & = & \{x : \neg A^\circ \mid x : A\ \mathit{false} \in \Delta\}. \end{array}$$

Note the change in the definition of $\Gamma^\circ$ which now assigns to $x$ type $\neg\neg A^\circ$. The translation of $\mathsf{callcc}\ x : A\ \mathit{false}.\ M$ and $\mathsf{throw}\ M\ \mathsf{to}\ x$ is the same as in the previous CPS translation and is omitted.

Case $\dfrac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathsf{K}} x : A}\ \mathsf{Hyp}$

$x^\circ =$

Case $\dfrac{\Gamma, x : A; \Delta \vdash_{\mathsf{K}} M : B}{\Gamma; \Delta \vdash_{\mathsf{K}} \lambda x{:}A.\, M : A \supset B}\ {\supset}\mathsf{I}$

$(\lambda x{:}A.\, M)^\circ =$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathsf{K}} N : A}{\Gamma; \Delta \vdash_{\mathsf{K}} M\ N : B}\ {\supset}\mathsf{E}$

$(M\ N)^\circ =$

## 5 Cut elimination [50 pts]

Consider the following fragment of the sequent calculus for propositional logic (with implication only):

$$\frac{}{\Gamma, A \longrightarrow A}\ \mathit{Init} \qquad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ {\supset}L \qquad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ {\supset}R$$

Give a proof of the admissibility of the cut rule:

**Theorem (Admissibility of the cut rule).** *If* $\Gamma \longrightarrow A$ *and* $\Gamma, A \longrightarrow C$, *then* $\Gamma \longrightarrow C$.

Begin your proof by stating how your nested induction proceeds, *e.g.*, "By nested induction on the structure of 1) $\cdots$; 2) $\cdots$; 3) $\cdots$." Your proof may use the weakening and contraction properties. In individual steps in your proof, please write the conclusion in the left side and the justification in the right side:

*conclusion* *justification*

Your proof should cover all possible cases.

## Work sheet

## Work sheet

**Work sheet**