CSE-433 Logic in Computer Science 2010
Final exam

- This is a closed-book exam. No other material is permitted.
- It consists of 5 problems worth a total of 210 points.
- There are 14 pages in this exam, including 3 work sheets.
- Try to use work sheets before writing your answers. Write your answers clearly and legibly.
- You have 3 hours for this exam.

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<td>210</td>
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1 Definitional equality and terms [30pts]

We write \( t = s \) to mean that terms \( t \) and \( s \) are definitionally equal.

\[
\begin{align*}
    t & \Rightarrow^* r \\
    s & \Rightarrow^* r \\
    t = s & \text{ DefEq}
\end{align*}
\]

In the introduction rule DefEq, the judgment \( t \Rightarrow^* r \) means that \( t \) reduces to \( r \) by zero or more \( \beta \)-reductions. Here we assume that \( \beta \)-reductions may be applied to subterms of the term being reduced.

**Question 1. [5 pts]** Definitional equality is transitive. Yes or no? Explain why. (-5 pts for a wrong answer)

**Question 2. [5 pts]** Consider terms for datatype nat:

\[
\text{term } t ::= \cdot \cdot \cdot | 0 | s(t) | \text{rec } f(t) \text{ of } f(0) \Rightarrow t | f(s(x)) \Rightarrow t
\]

Write two \( \beta \)-reductions for datatype nat:

\[
\Rightarrow^\beta
\]

\[
\Rightarrow^\beta
\]

**Question 3. [5 pts]** Define a function plus of datatype nat \( \rightarrow \) nat \( \rightarrow \) nat adding two natural numbers.

\[
\text{plus } =
\]

**Question 4. [5 pts]** plus 0 x = x holds. True or false? (-5 pts for a wrong answer)

**Question 5. [5 pts]** plus x 0 = x holds. True or false? (-5 pts for a wrong answer)

**Question 6. [5 pts]** plus s(x) y = s(plus x y) holds. True or false? (-5 pts for a wrong answer)
2 Local and global soundness and completeness [30pts]

Question 1. [5 pts] Explain local soundness in no more than 100 words.

Question 2. [10 pts] Explain global soundness in no more than 100 words.

Question 3. [5 pts] Explain local completeness in no more than 100 words.

Question 4. [10 pts] Explain global completeness in no more than 100 words.
3 First-order logic [50 pts]

Consider the natural deduction system for first-order logic (without datatypes). We use $t$ for terms, $x$ for term variables, and $a$ for parameters.

\[
\begin{align*}
[a/x]A \text{ true} & \quad \forall \text{I}^a \\
\forall x. A \text{ true} & \quad \forall \text{E} \\
[t/x]A \text{ true} & \quad \exists \text{I} \\
\exists x. A \text{ true} & \quad \exists \text{E}^a, w
\end{align*}
\]

**Question 1. [10 pts]**

Propose proof terms and their typing rules for first-order logic. Do not use hypothetical judgments with contexts.

**Question 2. [10 pts]**

Propose four inference rules for normal and neutral judgments ($A \uparrow$ and $A \downarrow$) in the natural deduction style. Do not use hypothetical judgments with contexts.
Question 3. [20 pts]
Propose four inference rules for sequents $\Gamma \vdash C$ in the sequent calculus.

Question 4. [10 pts]
Suppose that term variable $x$ is found in proposition $A$ and not in proposition $B$. That is, we have $[t/x]A \neq A$ and $[t/x]B = B$ in general for an arbitrary term $t$. Also recall that quantifiers have the lowest operator precedence, so, for example, we have $\forall x. A \supset B = \forall x. (A \supset B)$.

Give a proof of $(\exists x. A) \supset B \vdash \forall x. A \supset B$ using the rules that you have proposed. If not provable, state so.
4 Classical logic [50 pts]

In this problem, we use contexts $\Gamma$ and $\Delta$ defined as follows:

$$\Gamma ::= \cdot \mid \Gamma, x : A$$

$$\Delta ::= \cdot \mid \Delta, x : A \text{false}$$

We use $\Gamma; \Delta \vdash_K C \text{true}$ for typechecking proof terms in classical logic and $\Gamma \vdash_I M : C$ for typechecking proof terms in constructive logic.

Here are the rules Contra $\uparrow$ and Contra $\downarrow$ given in the Course Notes:

$$\Gamma; \Delta, A \text{false} \vdash_K A \text{true} \quad \text{Contra} \uparrow \quad \Gamma; \Delta, A \text{false} \vdash_K A \text{true} \quad \text{Contra} \downarrow$$

$$\Gamma; \Delta, A \text{false} \vdash_K C \text{true} \quad \text{Contra} \downarrow$$

**Question 1. [10 pts]** Use the rules Contra $\uparrow$ and Contra $\downarrow$ to show that the rule DNE is derivable.
**Question 2. [20 pts]** We use the following double-negation translation for the fragment of propositional logic with implication:

\[
P^\circ = P \quad (A \supset B)^\circ = A^\circ \supset \neg \neg B^\circ
\]

For each case below, complete the CPS translation \( M^\circ \) of a given proof term \( M \). Your translation should satisfy the invariant specified in the following theorem:

**Theorem.** If \( \Gamma; \Delta \vdash K M : C \), there exists a proof term \( M^\circ \) such that \( \Gamma^\circ, \neg \Delta^\circ \vdash I M^\circ : \neg \neg C^\circ \) where

\[
\Gamma^\circ = \{ x : A^\circ \mid x : A \in \Gamma \} \\
\neg \Delta^\circ = \{ x : \neg A^\circ \mid x : A \text{ false} \in \Delta \}.
\]

**Case** \( x : A \in \Gamma \)

\[
\Gamma; \Delta \vdash x : A \quad \text{Hyp}
\]

\[x^\circ = \]

**Case** \( \Gamma, x : A; \Delta \vdash K M : B \)

\[
\Gamma; \Delta \vdash \lambda x : A. M : A \supset B \quad \text{\$I}
\]

\[(\lambda x : A. M)^\circ = \]

**Case** \( \Gamma; \Delta \vdash K M : A \supset B \quad \Gamma; \Delta \vdash K N : A \)

\[
\Gamma; \Delta \vdash K MN : B \quad \text{\$E}
\]

\[(M N)^\circ = \]

**Case** \( \Gamma; \Delta, x : A \text{ false} \vdash K M : A \)

\[
\Gamma; \Delta \vdash \text{callcc } x : A \text{ false}. M : A \quad \text{Callcc}
\]

\[(\text{callcc } x : A \text{ false}. M)^\circ = \]

**Case** \( \Gamma; \Delta \vdash K M : A \quad x : A \text{ false} \in \Delta \)

\[
\Gamma; \Delta \vdash \text{throw } M \text{ to } x : C \quad \text{Throw}
\]

\[(\text{throw } M \text{ to } x)^\circ = \]
**Question 3. [20 pts]** The above CPS transformation corresponds to the call-by-value reduction strategy. To be specific, the translation of \( M \ N \) specifies that we finish evaluating \( N \) before we apply the function from \( M \) to the result of evaluating \( N \).

In this question, we will develop a variant of the CPS translation that corresponds to the call-by-name reduction strategy. We use the following double-negation translation (known as *Kolmogorov double negation translation*) in which \( (A \supset B)^\circ \) places \( \neg\neg \) before both \( A^\circ \) and \( B^\circ \):

\[
\begin{align*}
P^\circ &= P \\
(A \supset B)^\circ &= \neg\neg A^\circ \supset \neg\neg B^\circ
\end{align*}
\]

For each case below, complete the CPS translation \( M^\circ \) of a given proof term \( M \). Your translation should satisfy the invariant specified in the following theorem:

**Theorem.** If \( \Gamma; \Delta \vdash_K M : C \), there exists a proof term \( M^\circ \) such that \( \Gamma^\circ, \neg\Delta^\circ \vdash_I M^\circ : \neg\neg C^\circ \) where

\[
\begin{align*}
\Gamma^\circ &= \{ x : \neg\neg A^\circ \mid x : A \in \Gamma \} \\
\neg\Delta^\circ &= \{ x : \neg A^\circ \mid x : \text{false} \in \Delta \}.
\end{align*}
\]

Note the change in the definition of \( \Gamma^\circ \) which now assigns to \( x \) type \( \neg\neg A^\circ \). The translation of \( \text{callcc} \ x : A \text{ false}. M \) and \( \text{throw} M \) to \( x \) is the same as in the previous CPS translation and is omitted.

**Case** \( x : A \in \Gamma \)

\( \Gamma; \Delta \vdash_K \text{Hyp} \)

\[
x^\circ =
\]

**Case** \( \Gamma, x : A; \Delta \vdash_K M : B \)

\( \Gamma; \Delta \vdash_K \lambda x : A. M : A \supset B \supset I \)

\[
(\lambda x : A. M)^\circ =
\]

**Case** \( \Gamma; \Delta \vdash_K M : A \supset B \quad \Gamma; \Delta \vdash_K N : A \)

\( \Gamma; \Delta \vdash_K M \ N : B \supset E \)

\[
(M \ N)^\circ =
\]

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5 Cut elimination [50 pts]

Consider the following fragment of the sequent calculus for propositional logic (with implication only):

\[
\frac{\Gamma, A \rightarrow A}{\Gamma, A \rightarrow B} \quad \text{Init} \quad \frac{\Gamma, A \rightarrow B, B \rightarrow C}{\Gamma, A \rightarrow B} \quad \Gamma, A \rightarrow B \quad \rightarrow L \quad \frac{\Gamma, A \rightarrow B}{\Gamma, A \rightarrow C} \quad \rightarrow R
\]

Give a proof of the admissibility of the cut rule:

**Theorem (Admissibility of the cut rule).** If \(\Gamma \rightarrow A\) and \(\Gamma, A \rightarrow C\), then \(\Gamma \rightarrow C\).

Begin your proof by stating how your nested induction proceeds, e.g., “By nested induction on the structure of 1) \cdots; 2) \cdots; 3) \cdots.” Your proof may use the weakening and contraction properties. In individual steps in your proof, please write the conclusion in the left side and the justification in the right side:

<table>
<thead>
<tr>
<th>conclusion</th>
<th>justification</th>
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</thead>
<tbody>
<tr>
<td>(\Gamma \rightarrow C)</td>
<td>(1) (\Gamma \rightarrow A) and (\Gamma, A \rightarrow C)</td>
</tr>
</tbody>
</table>

Your proof should cover all possible cases.
Worksheet
Work sheet
Work sheet