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CSE-433 Logic in Computer Science 2011
Final exam

- This is a closed-book exam. No other material is permitted.
- It consists of 4 problems worth a total of 200 points.
- There are 10 pages in this exam, including 1 work sheet.
- Try to use work sheets before writing your answers. Write your answers clearly and legibly.
- You have 3 hours for this exam.

	Problem 1	Problem 2	Problem 3	Problem 4	Total
Score					
Max	50	50	50	50	200

1 Classical logic [50 pts]

In this problem, we use contexts Γ and Δ defined as follows:

$$\begin{array}{rcl} \Gamma & ::= & \cdot \mid \Gamma, x : A \\ \Delta & ::= & \cdot \mid \Delta, x : A\ \mathit{false} \end{array}$$

We use $\Gamma; \Delta \vdash_{\mathsf{K}} C\ \mathit{true}$ for judgments in classical logic, $\Gamma; \Delta \vdash_{\mathsf{K}} M : C$ for typechecking proof terms in classical logic, and $\Gamma \vdash_{\mathsf{I}} M : C$ for typechecking proof terms in constructive logic.

Here are the rules Contra↑ and Contra↓ given in the course notes:

$$\frac{\Gamma; \Delta, A\ \mathit{false} \vdash_{\mathsf{K}} A\ \mathit{true}}{\Gamma; \Delta \vdash_{\mathsf{K}} A\ \mathit{true}}\ \mathsf{Contra}\!\uparrow \qquad \frac{\Gamma; \Delta, A\ \mathit{false} \vdash_{\mathsf{K}} A\ \mathit{true}}{\Gamma; \Delta, A\ \mathit{false} \vdash_{\mathsf{K}} C\ \mathit{true}}\ \mathsf{Contra}\!\downarrow$$

Question 1. [10 pts] Show that the rule Peirce is derivable in classical logic.

$$\frac{}{\cdot; \cdot \vdash_{\mathsf{K}} ((A \supset B) \supset A) \supset A\ \mathit{true}}$$

Question 2. [20 pts] We use the following double-negation translation for the fragment of propositional logic with implication:

$$\begin{array}{rcl} P^\circ & = & P \\ (A \supset B)^\circ & = & A^\circ \supset \neg\neg B^\circ \end{array}$$

For each case below, complete the CPS translation M° of a given proof term M. Your translation should satisfy the invariant specified in the following theorem:

Theorem. *If* $\Gamma; \Delta \vdash_{\mathsf{K}} M : C$*, there exists a proof term* M° *such that* $\Gamma^\circ, \neg\Delta^\circ \vdash_{\mathsf{I}} M^\circ : \neg\neg C^\circ$ *where*

$$\begin{array}{rcl} \Gamma^\circ & = & \{x : A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ & = & \{x : \neg A^\circ \mid x : A \textit{ false} \in \Delta\}. \end{array}$$

Case $\dfrac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathsf{K}} x : A}$ Hyp

$x^\circ =$

Case $\dfrac{\Gamma, x : A; \Delta \vdash_{\mathsf{K}} M : B}{\Gamma; \Delta \vdash_{\mathsf{K}} \lambda x{:}A.\, M : A \supset B}$ $\supset$I

$(\lambda x{:}A.\, M)^\circ =$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathsf{K}} N : A}{\Gamma; \Delta \vdash_{\mathsf{K}} M\ N : B}$ $\supset$E

$(M\ N)^\circ =$

Case $\dfrac{\Gamma; \Delta, x : A \textit{ false} \vdash_{\mathsf{K}} M : A}{\Gamma; \Delta \vdash_{\mathsf{K}} \mathsf{callcc}\ x : A \textit{ false}.\, M : A}$ Callcc

$(\mathsf{callcc}\ x : A \textit{ false}.\, M)^\circ =$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \quad x : A \textit{ false} \in \Delta}{\Gamma; \Delta \vdash_{\mathsf{K}} \mathsf{throw}\ M\ \mathsf{to}\ x : C}$ Throw

$(\mathsf{throw}\ M\ \mathsf{to}\ x)^\circ =$

Question 3. [20 pts] The above CPS transformation corresponds to the call-by-value reduction strategy. To be specific, the translation of $M\ N$ specifies that we finish evaluating N before we apply the function from M to the result of evaluating N.

In this question, we will develop a variant of the CPS translation that corresponds to the call-by-name reduction strategy. We use the following double-negation translation (known as *Kolmogorov double negation translation*) in which $(A \supset B)^\circ$ places $\neg\neg$ before both A° and B°:

$$\begin{array}{rcl} P^\circ & = & P \\ (A \supset B)^\circ & = & \neg\neg A^\circ \supset \neg\neg B^\circ \end{array}$$

For each case below, complete the CPS translation M° of a given proof term M. Your translation should satisfy the invariant specified in the following theorem:

Theorem. *If $\Gamma; \Delta \vdash_{\mathsf{K}} M : C$, there exists a proof term M° such that $\Gamma^\circ, \neg\Delta^\circ \vdash_{\mathsf{I}} M^\circ : \neg\neg C^\circ$ where*

$$\begin{array}{rcl} \Gamma^\circ & = & \{x : \neg\neg A^\circ \mid x : A \in \Gamma\} \\ \neg\Delta^\circ & = & \{x : \neg A^\circ \mid x : A \; \mathit{false} \in \Delta\}. \end{array}$$

Note the change in the definition of Γ° which now assigns to x type $\neg\neg A^\circ$. The translation of $\mathsf{callcc}\ x : A\ \mathit{false}.\ M$ and $\mathsf{throw}\ M$ to x is the same as in the previous CPS translation and is omitted.

Case $\dfrac{x : A \in \Gamma}{\Gamma; \Delta \vdash_{\mathsf{K}} x : A}$ Hyp

$x^\circ =$

Case $\dfrac{\Gamma, x : A; \Delta \vdash_{\mathsf{K}} M : B}{\Gamma; \Delta \vdash_{\mathsf{K}} \lambda x{:}A.\, M : A \supset B}$ $\supset$I

$(\lambda x{:}A.\, M)^\circ =$

Case $\dfrac{\Gamma; \Delta \vdash_{\mathsf{K}} M : A \supset B \quad \Gamma; \Delta \vdash_{\mathsf{K}} N : A}{\Gamma; \Delta \vdash_{\mathsf{K}} M\ N : B}$ $\supset$E

$(M\ N)^\circ =$

2 Completeness of the sequent calculus [50 pts]

The sequent calculus for propositional logic is given as follows:

$$\frac{}{\Gamma, A \longrightarrow A}\ \mathit{Init} \quad \frac{\Gamma, A \wedge B, A \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L_L \quad \frac{\Gamma, A \wedge B, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L_R \quad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R$$

$$\frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L \quad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R$$

$$\frac{\Gamma, A \vee B, A \longrightarrow C \quad \Gamma, A \vee B, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_L \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_R$$

$$\frac{}{\Gamma \longrightarrow \top}\ \top R \quad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$

Show that the following rule $\supset L'$ is incomplete:

$$\frac{\Gamma \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L'$$

You should present a formula C such $\cdot \longrightarrow C$ is provable with the rule $\supset L$, but not with the rule $\supset L'$.

Question 1. [10 pts] What is such a formula C?

Question 2. [15 pts] Prove $\cdot \longrightarrow C$ using the rule $\supset L$.

Question 3. [15 pts] Show that $\cdot \longrightarrow C$ is not provable with the rule $\supset L'$.

Question 4. [10 pts] Explain what makes C a counter-example of the completeness of the rule $\supset L'$.

3 Backward proof search [50 pts]

Consider the sequent calculus for backward proof search:

$$\frac{\textit{atomic } P}{\Gamma, P \longrightarrow P}\ \textit{Init} \quad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L \quad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R \quad \frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L$$

$$\frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R \quad \frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_L \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_R$$

$$\frac{}{\Gamma \longrightarrow \top}\ \top R \quad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$

We reformulate the system using left propositions L, right propositions R, and two new sequents $\Delta; \Omega \Longrightarrow A; \cdot$ and $\Delta; \Omega \Longrightarrow \cdot; R$. In order to prove $\Gamma \longrightarrow A$, we try to prove $\cdot; \Gamma \Longrightarrow A; \cdot$ instead. Note that L and R are not invertible.

left proposition	L	$::=$	$P \mid A \supset A$	
right proposition	R	$::=$	$P \mid A \vee A \mid \bot$	
passive antecedents	Δ	$::=$	$\cdot \mid \Delta, L$	
active antecedents	Ω	$::=$	$\cdot \mid \Omega, A$	(ordered sequence)

Question 1. [15 pts] Write four inference rules for analyzing A in $\Delta; \Omega \Longrightarrow A; \cdot$:

Question 2. [20 pts] Write five inference rules for analyzing A in $\Delta; \Omega, A \Longrightarrow \cdot; R$:

Question 3. [10 pts] Write two inference rules for making a choice on R in $\Delta; \cdot \Longrightarrow \cdot; R$:

Question 4. [5 pts] Write an inference rule for making a choice on L in $\Delta, L; \cdot \Longrightarrow \cdot; R$:

4 Forward proof search [50 pts]

Consider the following fragment of the sequent calculus for propositional logic with $\wedge$ and $\supset$:

$$\frac{}{\Gamma, A \longrightarrow A}\ Init \quad \frac{\Gamma, A \wedge B, A \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L_L \quad \frac{\Gamma, A \wedge B, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L_R \quad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R$$

$$\frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L \quad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R$$

We use $\Gamma \Longrightarrow A$ in the forward sequent calculus, where we interpret Γ as a set. For example, Γ, A is equal to Γ if A is already in Γ. We also write $\Gamma \cup \Gamma'$ for the union of Γ and Γ'.

The two invariants to maintain are:

- Soundness: If $\Gamma \Longrightarrow C$, then $\Gamma \longrightarrow C$.
- Completeness: If $\Gamma \longrightarrow C$, then $\Gamma' \Longrightarrow C$ for some $\Gamma' \subset \Gamma$.

Write inference rules for the forward sequent calculus. You need a total of 7 inference rules.

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Work sheet