CSE-433 Logic in Computer Science 2011
Final exam

- This is a closed-book exam. No other material is permitted.
- It consists of 4 problems worth a total of 200 points.
- There are 10 pages in this exam, including 1 work sheet.
- Try to use work sheets before writing your answers. Write your answers clearly and legibly.
- You have 3 hours for this exam.

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1 Classical logic [50 pts]

In this problem, we use contexts \( \Gamma \) and \( \Delta \) defined as follows:

\[
\begin{align*}
\Gamma & ::= \cdot \mid \Gamma, x : A \\
\Delta & ::= \cdot \mid \Delta, x : A \text{false}
\end{align*}
\]

We use \( \Gamma; \Delta \vdash K \text{true} \) for judgments in classical logic, \( \Gamma; \Delta \vdash M : C \) for typechecking proof terms in classical logic, and \( \Gamma \vdash I \) \( M : C \) for typechecking proof terms in constructive logic.

Here are the rules Contra↑ and Contra↓ given in the course notes:

\[
\Gamma; \Delta, A \text{false} \vdash K \text{true} \quad \text{Contra↑}
\]

\[
\Gamma; \Delta \vdash K A \text{true} \quad \Gamma; \Delta, A \text{false} \vdash K A \text{true} \quad \text{Contra↓}
\]

Question 1. [10 pts] Show that the rule Peirce is derivable in classical logic.

\[
\vdash \vdash (A \lor B) \lor A \vdash A \text{true}
\]
**Question 2. [20 pts]** We use the following double-negation translation for the fragment of propositional logic with implication:

\[ P^\circ = P \]
\[ (A \lor B)^{\circ} = A^\circ \lor B^\circ \]

For each case below, complete the CPS translation \( M^\circ \) of a given proof term \( M \). Your translation should satisfy the invariant specified in the following theorem:

**Theorem.** If \( \Gamma; \Delta \vdash_K M : C \), there exists a proof term \( M^\circ \) such that \( \Gamma^\circ, \neg \Delta^\circ \vdash_I M^\circ : \neg \neg C^\circ \) where

\[ \Gamma^\circ = \{ x : A^\circ \mid x : A \in \Gamma \} \]
\[ \neg \Delta^\circ = \{ x : \neg A^\circ \mid x : A \text{ false} \in \Delta \} \]

Case \( x : A \in \Gamma \)

\[ \Gamma; \Delta \vdash_K x : A \text{ Hyp} \]

\( x^\circ = \)

Case \( \Gamma, x : A; \Delta \vdash_K M : B \)

\[ \Gamma; \Delta \vdash_K \lambda x: A. M : A \lor B \ \triangleright I \]

\( (\lambda x: A. M)^\circ = \)

Case \( \Gamma; \Delta \vdash_K M : A \lor B \)

\[ \Gamma; \Delta \vdash_K N : A \]

\[ \Gamma; \Delta \vdash_K M \lor N : B \ \triangleright E \]

\( (M \lor N)^\circ = \)

Case \( \Gamma; \Delta, x : A \text{ false} \vdash_K M : A \)

\[ \Gamma; \Delta \vdash_K \text{callcc } x : A \text{ false}. M : A \text{ Callcc} \]

\( (\text{callcc } x : A \text{ false}. M)^\circ = \)

Case \( \Gamma; \Delta \vdash_K M : A \)

\[ x : A \text{ false} \in \Delta \]

\[ \Gamma; \Delta \vdash_K \text{throw } M \text{ to } x : C \ \text{ Throw} \]

\( (\text{throw } M \text{ to } x)^\circ = \)
**Question 3. [20 pts]** The above CPS transformation corresponds to the call-by-value reduction strategy. To be specific, the translation of \( M N \) specifies that we finish evaluating \( N \) before we apply the function from \( M \) to the result of evaluating \( N \).

In this question, we will develop a variant of the CPS translation that corresponds to the call-by-name reduction strategy. We use the following double-negation translation (known as Kolmogorov double negation translation) in which \((A \supset B)^{\circ}\) places \(\neg\) before both \(A^{\circ}\) and \(B^{\circ}\):

\[
P^{\circ} = P \\
(A \supset B)^{\circ} = \neg\neg A^{\circ} \supset \neg\neg B^{\circ}
\]

For each case below, complete the CPS translation \(M^{\circ}\) of a given proof term \(M\). Your translation should satisfy the invariant specified in the following theorem:

**Theorem.** If \(\Gamma; \Delta \vdash_K M : C\), there exists a proof term \(M^{\circ}\) such that \(\Gamma^{\circ}, \neg\Delta^{\circ} \vdash I M^{\circ} : \neg\neg C^{\circ}\) where

\[
\Gamma^{\circ} = \{ x : \neg\neg A^{\circ} \mid x : A \in \Gamma \} \\
\neg\Delta^{\circ} = \{ x : \neg\neg A^{\circ} \mid x : A false \in \Delta \}.
\]

Note the change in the definition of \(\Gamma^{\circ}\) which now assigns to \(x\) type \(\neg\neg A^{\circ}\). The translation of \(\text{callcc } x : A \text{ false. } M\) and \(\text{throw } M \text{ to } x\) is the same as in the previous CPS translation and is omitted.

**Case** \(\dfrac{x : A \in \Gamma}{\Gamma; \Delta \vdash_K x : A \text{ Hyp}}\)

\(x^{\circ} = \)

**Case** \(\dfrac{\Gamma; x : A; \Delta \vdash_K M : B}{\Gamma; \Delta \vdash_K \lambda x : A. M : A \supset B \supset I} \)

\((\lambda x : A. M)^{\circ} = \)

**Case** \(\dfrac{\Gamma; \Delta \vdash_K M : A \supset B \quad \Gamma; \Delta \vdash_K N : A}{\Gamma; \Delta \vdash_K M \ N : B \supset E} \)

\((M \ N)^{\circ} = \)
2 Completeness of the sequent calculus [50 pts]

The sequent calculus for propositional logic is given as follows:

\[ \frac{}{\Gamma, A \rightarrow A} \quad \text{Init} \]
\[ \frac{\Gamma \rightarrow A, \Gamma \rightarrow B, A \rightarrow C}{\Gamma, A \land B \rightarrow C} \quad \land L \]
\[ \frac{\Gamma, A \land B \rightarrow C}{\Gamma, A \rightarrow C} \quad \land R \]
\[ \frac{\Gamma, A 
abla B \rightarrow A, \Gamma, A 
abla B, B \rightarrow C}{\Gamma, A 
abla B \rightarrow C} \quad 
abla L \]
\[ \frac{\Gamma, A \rightarrow B}{\Gamma 
abla B \rightarrow C} \quad 
abla R \]
\[ \frac{\Gamma, A \rightarrow B, A \nabla B \rightarrow C}{\Gamma 
abla B \rightarrow C} \quad \lor L \]
\[ \frac{\Gamma, A \nabla B \rightarrow C}{\Gamma, A \rightarrow B} \quad \lor R \]
\[ \frac{\Gamma, A \rightarrow B, A \nabla B \rightarrow C}{\Gamma \rightarrow C} \quad \lor R \]
\[ \frac{}{\Gamma \rightarrow C} \quad \top R \]
\[ \frac{}{\Gamma, \bot \rightarrow C} \quad \bot L \]

Show that the following rule \( \supset L' \) is incomplete:

\[ \frac{\Gamma \rightarrow A, \Gamma \supset B, A \rightarrow C}{\Gamma, A \supset B \rightarrow C} \quad \supset L' \]

You should present a formula \( C \) such that \( \cdot \rightarrow C \) is provable with the rule \( \supset L \), but not with the rule \( \supset L' \).

**Question 1.** [10 pts] What is such a formula \( C \)?

**Question 2.** [15 pts] Prove \( \cdot \rightarrow C \) using the rule \( \supset L \).
Question 3. [15 pts] Show that $\cdot \rightarrow C$ is not provable with the rule $\supset L'$.

Question 4. [10 pts] Explain what makes $C$ a counter-example of the completeness of the rule $\supset L'$. 
3 Backward proof search [50 pts]

Consider the sequent calculus for backward proof search:

\[
\begin{align*}
\text{atomic } P & \quad \Gamma, P \rightarrow P & \text{Init} \\
\Gamma, A, B \rightarrow C & \quad \Gamma, A \land B \rightarrow C & \land L \\
\Gamma \rightarrow A & \quad \Gamma \rightarrow A \land B & \land R \\
\Gamma, A \supset B \rightarrow A & \quad \Gamma, B \rightarrow C & \supset L \\
\Gamma, A \rightarrow B & \quad \Gamma \rightarrow A \supset B & \supset R \\
\Gamma \rightarrow A \land B & \quad \Gamma \rightarrow A \lor B & \lor L \\
\Gamma \rightarrow A & \quad \Gamma \rightarrow A \lor B & \lor R_L \\
\Gamma \rightarrow A \lor B & \quad \Gamma \rightarrow A \lor B & \lor R_R \\
\Gamma \rightarrow \top & \quad \Gamma, \bot \rightarrow C & \bot L
\end{align*}
\]

We reformulate the system using left propositions \(L\), right propositions \(R\), and two new sequents \(\Delta; \Omega \Rightarrow A; \cdot\) and \(\Delta; \Omega \Rightarrow \cdot; R\). In order to prove \(\Gamma \rightarrow A\), we try to prove \(\cdot; \Gamma \Rightarrow A; \cdot\) instead. Note that \(L\) and \(R\) are not invertible.

- left proposition \(L \::=\ P \mid A \supset A\)
- right proposition \(R \::=\ P \mid A \lor A \mid \bot\)
- passive antecedents \(\Delta \::=\ \cdot \mid \Delta, L\)
- active antecedents \(\Omega \::=\ \cdot \mid \Omega, A\) (ordered sequence)

**Question 1. [15 pts]** Write four inference rules for analyzing \(A\) in \(\Delta; \Omega \Rightarrow A; \cdot\)
Question 2. [20 pts] Write five inference rules for analyzing $A$ in $\Delta; \Omega, A \implies \cdot; R$.

Question 3. [10 pts] Write two inference rules for making a choice on $R$ in $\Delta; \cdot \implies \cdot; R$.

Question 4. [5 pts] Write an inference rule for making a choice on $L$ in $\Delta; L; \cdot \implies \cdot; R$. 

4 Forward proof search [50 pts]

Consider the following fragment of the sequent calculus for propositional logic with $\land$ and $\supset$:

\[
\frac{\Gamma, A \rightarrow A}{\text{Init}} \quad \frac{\Gamma, A \land B, A \rightarrow C}{\land_L} \quad \frac{\Gamma, A \land B, B \rightarrow C}{\land_R} \\
\frac{\Gamma, A \land B \rightarrow C}{\land_L} \quad \frac{\Gamma, A \land B \rightarrow C}{\land_R} \quad \frac{\Gamma \rightarrow A \Gamma \rightarrow B}{\land_R}
\]

We use $\Gamma \Rightarrow A$ in the forward sequent calculus, where we interpret $\Gamma$ as a set. For example, $\Gamma, A$ is equal to $\Gamma$ if $A$ is already in $\Gamma$. We also write $\Gamma \cup \Gamma'$ for the union of $\Gamma$ and $\Gamma'$.

The two invariants to maintain are:

- **Soundness**: If $\Gamma \Rightarrow C$, then $\Gamma \rightarrow C$.
- **Completeness**: If $\Gamma \rightarrow C$, then $\Gamma' \Rightarrow C$ for some $\Gamma' \subset \Gamma$.

Write inference rules for the forward sequent calculus. You need a total of 7 inference rules.
Work sheet