CSE-433 Assignment - *Decision Procedure for Constructive Propositional Logic*

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- For this assignment, you may discuss implementation ideas and techniques with your classmates. The code you submit, however, must be your own work.

Please download `prover.zip` which contains `prop.ml`, `prover.ml`, and `sources.cm`. `prop.ml` contains the library code for this assignment and `test.ml` contains a test fuctor `F`. Your goal is to implement the function `decide` in the modules `Backward` and `Forward`.

1 Backward prover

Read Pfenning Chapter 4.1 and implement a backward prover for constructive propositional logic.

The original sequent calculus is given as follows. Note that we apply the rule *Init* only to atomic formulas.

$$\frac{\textit{atomic } P}{\Gamma, P \longrightarrow P}\ \textit{Init} \quad \frac{\Gamma, A \wedge B, A \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L_L \quad \frac{\Gamma, A \wedge B, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L_R \quad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R$$

$$\frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, A \supset B, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L \quad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R$$

$$\frac{\Gamma, A \vee B, A \longrightarrow C \quad \Gamma, A \vee B, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_L \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_R$$

$$\frac{}{\Gamma \longrightarrow \top}\ \top R \quad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$

We combine the two rules $\wedge L_L$ and $\wedge L_R$, and remove unnecessary formulas in the premises of the rules $\supset L$ and $\vee L$. Axioms are *Init*, $\top R$, and $\bot L$. Invertible rules are $\wedge L$, $\wedge R$, $\supset R$, and $\vee L$. Non-invertible rules are $\supset L$, $\vee R_L$, and $\vee R_R$.

$$\frac{\textit{atomic } P}{\Gamma, P \longrightarrow P}\ \textit{Init} \quad \frac{\Gamma, A, B \longrightarrow C}{\Gamma, A \wedge B \longrightarrow C}\ \wedge L \quad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B}\ \wedge R$$

$$\frac{\Gamma, A \supset B \longrightarrow A \quad \Gamma, B \longrightarrow C}{\Gamma, A \supset B \longrightarrow C}\ \supset L \quad \frac{\Gamma, A \longrightarrow B}{\Gamma \longrightarrow A \supset B}\ \supset R$$

$$\frac{\Gamma, A \longrightarrow C \quad \Gamma, B \longrightarrow C}{\Gamma, A \vee B \longrightarrow C}\ \vee L \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow A \vee B}\ \vee R_L \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow A \vee B}\ \vee R_R$$

$$\frac{}{\Gamma \longrightarrow \top}\ \top R \quad \frac{}{\Gamma, \bot \longrightarrow C}\ \bot L$$

Now we reformulate the system using left propositions L, right propositions R, and two new sequents $\Delta; \Omega \Longrightarrow A; \cdot$ and $\Delta; \Omega \Longrightarrow \cdot; R$. The idea is that in order to prove $\Gamma \longrightarrow A$, we try to prove $\cdot; \Gamma \Longrightarrow A; \cdot$

instead.

left proposition	L	$::=$	$P \mid A \supset A$
right proposition	R	$::=$	$P \mid A \vee A \mid \bot$
passive antecedents	Δ	$::=$	$\cdot \mid \Delta, L$
active antecedents	Ω	$::=$	$\cdot \mid \Omega, A$ (ordered sequence)

The inference rules for analyzing A in $\Delta; \Omega \Longrightarrow A; \cdot$ are:

$$\frac{\Delta; \Omega \Longrightarrow A; \cdot \quad \Delta; \Omega \Longrightarrow B; \cdot}{\Delta; \Omega \Longrightarrow A \wedge B; \cdot} \wedge R \qquad \frac{}{\Delta; \Omega \Longrightarrow \top; \cdot} \top R$$

$$\frac{\Delta; \Omega, A \Longrightarrow B; \cdot}{\Delta; \Omega \Longrightarrow A \supset B; \cdot} \supset R \qquad \frac{\Delta; \Omega \Longrightarrow \cdot; R}{\Delta; \Omega \Longrightarrow R; \cdot} RR$$

The inference rules for analyzing A in $\Delta; \Omega, A \Longrightarrow \cdot; R$ are:

$$\frac{\Delta; \Omega, A, B \Longrightarrow \cdot; R}{\Delta; \Omega, A \wedge B \Longrightarrow \cdot; R} \wedge L \qquad \frac{\Delta; \Omega \Longrightarrow \cdot; R}{\Delta; \Omega, \top \Longrightarrow \cdot; R} \top L$$

$$\frac{\Delta; \Omega, A \Longrightarrow \cdot; R \quad \Delta; \Omega, B \Longrightarrow \cdot; R}{\Delta; \Omega, A \vee B \Longrightarrow \cdot; R} \vee L \qquad \frac{}{\Delta; \Omega, \bot \Longrightarrow \cdot; R} \bot L$$

$$\frac{\Delta, L; \Omega \Longrightarrow \cdot; R}{\Delta; \Omega, L \Longrightarrow \cdot; R} LL$$

The inference rules for making a choice on R in $\Delta; \cdot \Longrightarrow \cdot; R$ are:

$$\frac{\Delta; \cdot \Longrightarrow A; \cdot}{\Delta; \cdot \Longrightarrow \cdot; A \vee B} \vee R_L \qquad \frac{\Delta; \cdot \Longrightarrow B; \cdot}{\Delta; \cdot \Longrightarrow \cdot; A \vee B} \vee R_R$$

The inference rules for making a choice on L in $\Delta, L; \cdot \Longrightarrow \cdot; R$ are:

$$\frac{\Delta, A \supset B; \cdot \Longrightarrow A; \cdot \quad \Delta; B \Longrightarrow \cdot; R}{\Delta, A \supset B; \cdot \Longrightarrow \cdot; R} \supset L$$

The inference rule for initial sequents is:

$$\frac{\textit{atomic } P}{\Delta, P; \cdot \Longrightarrow \cdot; P} \textit{Init}$$

Design your prover so that it terminates on all input propositions.

2 Forward prover

Read Pfenning Chapter 5 and implement a forward prover for constructive propositional logic.