Chapter 1

First-Order Logic

This chapter develops *first-order logic, i.e.,* logic with universal and existential quantifications. Developing first-order logic is the first step toward a practical reasoning system which inevitably demands an apparatus for expressing that a given property holds *for all, or* $\forall$, objects or that there *exists, or* $\exists$, a certain object satisfying a given property. Here we deal with pure first-order logic which does not stipulate a particular class of objects. Later we will enrich it in such a way that we can express properties of specific classes of objects such as natural numbers, trees, or boolean values.

1.1 Terms

In propositional logic, expressing properties of objects under consideration requires us to define propositional constants which denote atomic propositions. For example, in order to express that 1 is equal to 1 itself, we would need a propositional constant Eq_1 denoting an atomic proposition *'1 is equal to 1.'* While logical connectives provide us with an elegant mechanism for reasoning about such atomic propositions, the need for a separate propositional constant for each atomic proposition makes propositional logic too limited in its expressive power. For example, in order to express that every natural number is equal to itself, we would have to define an infinite array of propositional constants Eq_i denoting *'i is equal to i.'*

First-order logic replaces propositional constants in propositional logic by *predicates*. A predicate may have arguments and expresses a relation between its arguments. (For this reason, first-order logic is also called *predicate logic*.) For example, we can define a predicate Eq so that $Eq(t_1, t_2)$ denotes a proposition *'t_1 is equal to t_2.'* Here the predicate Eq has two arguments t_1 and t_2 and expresses an equality between t_1 and t_2. The arguments t_1 and t_2 are called *terms* in first-order logic and may be interpreted as particular mathematical objects (such as natural numbers). Thus first-order logic is a system in which we use predicates to express properties of terms.

Note that first-order logic itself does not enforce a specific way of interpreting terms. As an example, consider two terms $\mathbf{0}$ and $\mathsf{s}(\mathbf{0})$. As usual, we could interpret $\mathbf{0}$ as zero and $\mathsf{s}(\mathbf{0})$ as the successor of zero, but such an interpretation is just a specific way of assigning mathematical objects to terms. Thus it is also fine to interpret $\mathbf{0}$ as the natural number one and or $\mathsf{s}(\mathbf{0})$ as the predecessor of one. In general, we do not formalize how to relate terms to mathematical objects, and first-order logic in our discussion (which is based on proof theory) deals only with terms and not with their interpretations. Thus predicates directly express properties of uninterpreted terms.

Formally we define terms as follows:

$$\text{term} \quad t, s \quad ::= \quad x \mid y \mid \cdots \mid a \mid b \mid \cdots \mid f(t_1, \cdots, t_n) \mid c$$

$x, y, \cdots$ are called *term variables* which range over the set of all terms. We may substitute terms for term variables and we write $[s/x]t$ for the result of substituting s for x in t. $a, b, \cdots$ are called *parameters* and denote arbitrary/unspecified terms about which we can make no assumption. The difference between term variables and parameters is that a term variable is just a placeholder for another term whereas a parameter is understood as an arbitrary term about which nothing is known. (We will see the use of parameters in inference rules for first-order logic.)

f is called a *function symbol* and has zero or more arguments. We write $f(t_1, \cdots, t_n)$ for a term where f is a function symbol of arity n and t_1, $\cdots$, t_n are its arguments. A *constant* c is a function symbol of zero arity; that is, c is an abbreviation of $c()$. Note that although it is usually interpreted as a function in the mathematical sense, a function symbol f is *not* a function because $f(t_1, \cdots, t_n)$ is a term in itself and does not reduce to another term. For example, $\mathrm{s}(\mathbf{0})$, which comprises of a function symbol s and its argument $\mathbf{0}$, does not reduce to another term, say $\mathbf{1}$, because it is a term in itself.

Now we can define a set of terms by specifying function symbols with their arities. Here are a few examples:

- To obtain terms for natural numbers, we use a constant $\mathbf{0}$ for zero and a function symbol s of arity one to be interpreted as the successor function.
- To obtain terms for boolean values, we use two constants **true** and **false**.
- To obtain terms for binary trees, we use a constant **leaf** for leaf nodes and a function symbol **node** of arity two for inner nodes.

Terms are not to be confused with proof terms. Terms can represent any kinds of objects (*e.g.*, natural numbers, boolean values, student names, *etc.*) whereas proof terms represent proofs in logic. For example, we can say that a proof term $\lambda x\!:\!A.\,x$ represents a proof of $A \supset A$, but it makes no sense to judge the truth or falsehood of a term $\mathrm{s}(\mathbf{0})$.

1.2 Propositions in first-order logic

In addition to logical connectives from propositional logic, first-order logic uses predicates and two forms of quantifications over terms. An inductive definition of propositions is given as follows:

$$\text{proposition} \quad A \quad ::= \quad P(t_1, \cdots, t_n) \mid \cdots \mid \forall x.A \mid \exists x.A$$

Alternatively we may use three new formation rules:

$$\frac{}{P(t_1, \cdots, t_n)\ \mathit{prop}}\ P\mathsf{F} \qquad \frac{A\ \mathit{prop}}{\forall x.A\ \mathit{prop}}\ \forall\mathsf{F} \qquad \frac{A\ \mathit{prop}}{\exists x.A\ \mathit{prop}}\ \exists\mathsf{F}$$

P is called a *predicate symbol*. A predicate $P(t_1, \cdots, t_n)$ is a proposition that expresses a certain relation between terms t_1, $\cdots$, t_n. For example, we may use $\mathit{Nat}(t)$ to mean that term t is a natural number, or $\mathit{Eq}(t_1, t_2)$ to mean that terms t_1 and t_2 are equal. A propositional constant P is a predicate symbol of zero arity; that is, P is an abbreviation of $P()$.

$\forall x.A$ uses a *universal quantifier* $\forall$ to introduce a term variable x. Roughly speaking, the truth of $\forall x.A$ means that A is true for "every" term x. $\exists x.A$ uses an *existential quantifier* $\exists$ to introduce a term variable x. Roughly speaking, the truth of $\exists x.A$ means that we can present "some" term x for which A is true. Quantifiers $\forall$ and $\exists$ have the lowest operator precedence. For example, $\forall x.A \supset B$ is understood as $\forall x.(A \supset B)$; similarly $\exists x.A \supset B$ is understood as $\exists x.(A \supset B)$.

As quantifiers introduce term variables, there arises a need for substitutions for term variables in propositions or proofs. We write $[t/x]A$ for the result of substituting t for x in proposition A. Similarly we write $[t/x]\mathcal{D}$ for the result of substituting t for x throughout proof $\mathcal{D}$. Extending substitutions for term variables, we write $[t/a]A$ and $[t/a]\mathcal{D}$ for the result of substituting t for parameter a in A and $\mathcal{D}$, respectively. These substitutions for term variables and parameters are considerably simpler to define than substitutions in the simply-typed λ-calculus because variable captures never occur in first-order logic. That is, in a substitution $[t/x]A$ or $[t/x]\mathcal{D}$, term t is always closed and contains no free term variables.

1.3 Universal quantification

A universal quantification $\forall x.A$ is true if A is true for every term x. For example, given that $\mathbf{0}$, $\mathrm{s}(\mathbf{0})$, $\mathrm{s}(\mathrm{s}(\mathbf{0}))$, $\cdots$ constitute the set of terms, we can deduce $\forall x.\mathit{Eq}(x, x)\ \mathit{true}$ if $\mathit{Eq}(\mathbf{0}, \mathbf{0})\ \mathit{true}$, $\mathit{Eq}(\mathrm{s}(\mathbf{0}), \mathrm{s}(\mathbf{0}))\ \mathit{true}$, $\mathit{Eq}(\mathrm{s}(\mathrm{s}(\mathbf{0})), \mathrm{s}(\mathrm{s}(\mathbf{0})))\ \mathit{true}$, $\cdots$ are all provable. Hence it helps to think of $\forall x.A$ as an infinite conjunction

$$[t_1/x]A \wedge [t_2/x]A \wedge \cdots \wedge [t_i/x]A \wedge \cdots$$

where $t_1, t_2, \cdots, t_i, \cdots$ enumerate all terms.

The inference rules for universal quantifications are given as follows:

$$\frac{[a/x]A\ \mathit{true}}{\forall x.A\ \mathit{true}}\ \forall\mathsf{I}^a \qquad \frac{\forall x.A\ \mathit{true}}{[t/x]A\ \mathit{true}}\ \forall\mathsf{E}$$

In the rule $\forall\mathsf{I}^a$, parameter a denotes an arbitrary term about which we can make no assumption. Thus we may read $[a/x]A\ \mathit{true}$ as a shorthand for a sequence of judgments

$$[t_1/x]A\ \mathit{true} \quad [t_2/x]A\ \mathit{true} \quad \cdots \quad [t_i/x]A\ \mathit{true} \quad \cdots$$

where $t_1, t_2, \cdots, t_i, \cdots$ enumerate all terms. In the rule $\forall\mathsf{E}$, t can be any term — a constant, a function symbol, a term variable, or even an existing parameter. We justify the rule $\forall\mathsf{E}$ by reading $\forall x.A\ \mathit{true}$ as

$$[t_1/x]A \wedge [t_2/x]A \wedge \cdots \wedge [t_i/x]A \wedge \cdots\ \mathit{true}$$

where $t_1, t_2, \cdots, t_i, \cdots$ enumerate all terms.

It is important that in the rule $\forall\mathsf{I}^a$, parameter a must be fresh and not found in any undischarged hypothesis. For example, a proof of $\forall x.\mathit{Nat}(x)\ \mathit{true}$ introducing a fresh parameter a must not contain any hypothesis of the form $\overline{P(a)}$, which is an assumption on an arbitrary term about which we can make no assumption! The presence of such a hypothesis implies that parameter a is already declared elsewhere and thus cannot be interpreted as an arbitrary term. The following example, which tries to prove that y is a natural number whenever x is a natural number, shows that using the same parameter twice in difference instances of the rule $\forall\mathsf{I}$ results in a wrong proof:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\mathit{Nat}(a)\ \mathit{true}}^{w}}{\forall x.\mathit{Nat}(x)\ \mathit{true}}\ \forall\mathsf{I}^a\ \textit{(wrong)}}{\mathit{Nat}(b)\ \mathit{true}}\ \forall\mathsf{E}}{\mathit{Nat}(a) \supset \mathit{Nat}(b)\ \mathit{true}}\ \supset\mathsf{I}^w}{\forall y.\mathit{Nat}(a) \supset \mathit{Nat}(y)\ \mathit{true}}\ \forall\mathsf{I}^b}{\forall x.\forall y.\mathit{Nat}(x) \supset \mathit{Nat}(y)\ \mathit{true}}\ \forall\mathsf{I}^a$$

Here is an example of a proof involving universal quantifiers where we exploit $[a/x](A \wedge B) = [a/x]A \wedge [a/x]B$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\forall x.A \wedge B\ \mathit{true}}^{w}}{[a/x](A \wedge B)\ \mathit{true}}\ \forall\mathsf{E}}{[a/x]A\ \mathit{true}}\ \wedge\mathsf{E_L}}{\forall x.A\ \mathit{true}}\ \forall\mathsf{I}^a \qquad \cfrac{\cfrac{\cfrac{\cfrac{}{\forall x.A \wedge B\ \mathit{true}}^{w}}{[a/x](A \wedge B)\ \mathit{true}}\ \forall\mathsf{E}}{[a/x]B\ \mathit{true}}\ \wedge\mathsf{E_R}}{\forall x.B\ \mathit{true}}\ \forall\mathsf{I}^a}{(\forall x.A) \wedge (\forall x.B)\ \mathit{true}}\ \wedge\mathsf{I}}{(\forall x.A \wedge B) \supset (\forall x.A) \wedge (\forall x.B)\ \mathit{true}}\ \supset\mathsf{I}^w$$

1.4 Existential quantification

An existential quantification $\exists x.A$ is true if there exists a term x satisfying A. For example, if $\mathit{Eq}(\mathbf{0}, \mathbf{0})\ \mathit{true}$ is provable, we can deduce $\exists x.\mathit{Eq}(x, x)$ because substituting a concrete term $\mathbf{0}$ for x makes $\mathit{Eq}(x, x)\ \mathit{true}$ provable. Hence it helps to think of $\exists x.A$ as an infinite disjunction

$$[t_1/x]A \vee [t_2/x]A \vee \cdots \wedge [t_i/x]A \vee \cdots$$

where $t_1, t_2, \cdots, t_i, \cdots$ enumerate all terms.

The inference rules for existential quantifications are given as follows:

$$\frac{[t/x]A\ \mathit{true}}{\exists x.A\ \mathit{true}}\ \exists\mathsf{I} \qquad \frac{\exists x.A\ \mathit{true} \qquad \begin{matrix} \overline{[a/x]A\ \mathit{true}}^{\,w} \\ \vdots \\ C\ \mathit{true} \end{matrix}}{C\ \mathit{true}}\ \exists\mathsf{E}^{a,w}$$

The rule ∃I says that we prove $\exists x.A\ \mathit{true}$ by presenting a concrete term, or a *witness*, t such that $[t/x]A\ \mathit{true}$ is provable. We justify the rule ∃I by reading $\exists x.A\ \mathit{true}$ as

$$[t_1/x]A \vee [t_2/x]A \vee \cdots \vee [t_i/x]A \vee \cdots\ \mathit{true}$$

where $t_1,\ t_2,\ \cdots,\ t_i,\ \cdots$ enumerate all terms and $t_i = t$ holds. In the rule $\exists\mathsf{E}^{a,w}$, we annotate the hypothesis $\overline{[a/x]A\ \mathit{true}}$ with label w. We also introduce a fresh parameter a because the witness for the proof of $\exists x.A\ \mathit{true}$ is unknown and thus we cannot make any assumption about it. Thus we may read $\begin{array}{c}\overline{[a/x]A\ \mathit{true}}^{\,w}\\ \vdots\\ C\ \mathit{true}\end{array}$ as a shorthand for a sequence of hypothetical proofs

$$\begin{array}{ccccc}\overline{[t_1/x]A\ \mathit{true}}^{\,w} & \overline{[t_2/x]A\ \mathit{true}}^{\,w} & & \overline{[t_i/x]A\ \mathit{true}}^{\,w} & \\ \vdots & \vdots & \cdots & \vdots & \cdots \\ C\ \mathit{true} & C\ \mathit{true} & & C\ \mathit{true} & \end{array}$$

where $t_1, t_2, \cdots, t_i, \cdots$ enumerate all terms.

In the rule $\exists\mathsf{E}^{a,w}$, parameter a must be fresh and not found in proposition A or any undischarged hypothesis. In particular, it must not be found in proposition C. Otherwise the rule ends up with a conclusion that makes too strong an assumption about the witness, namely that the witness can be an arbitrary term! For example, the following proof exploits a proof of $\exists x.Nat(x) \wedge Eq(x, \mathbf{0})\ \mathit{true}$ to draw a (nonsensical) conclusion that an arbitrary term is equal to a natural number **0**, as it allows parameter a to appear in the conclusion:

$$\dfrac{\begin{array}{c}\vdots\\ \exists x.Nat(x) \wedge Eq(x,\mathbf{0})\ \mathit{true}\end{array} \quad \dfrac{\overline{Nat(a) \wedge Eq(a,\mathbf{0})\ \mathit{true}}^{\,w}}{Eq(a,\mathbf{0})\ \mathit{true}}\wedge\mathsf{E_R}}{Eq(a,\mathbf{0})\ \mathit{true}}\exists\mathsf{E}^{a,w}$$

In essence, the rule $\exists\mathsf{E}^{a,w}$ introduces parameter a in the course of proving $C\ \mathit{true}$ after fixing proposition C, which implies that C is oblivious to a.

An important aspect of the rule ∃I is that in order to prove $\exists x.A\ \mathit{true}$, it is not enough to show that there only "exists" a witness x satisfying A without actually knowing what it is. The necessity of such a witness is indeed a distinguishing feature of constructive logic. In contrast, a proof of $\exists x.A\ \mathit{true}$ in classical logic only needs to show that there exists a term t, *which may or may not be known,* such that $[t/x]A\ \mathit{true}$ is provable. In other words, a proof of $\exists x.A\ \mathit{true}$ essentially shows that it cannot happen that there exists no term t such that $[t/x]A\ \mathit{true}$ is provable. As a consequence, $\exists x.A$ is no different from $\neg\forall x.\neg A$ in classical logic.

To better understand the nature of existential quantifications in constructive logic, let us consider a few examples. First $\exists x.\neg A \supset \neg\forall x.A\ \mathit{true}$ is provable. Intuitively a proof of $\exists x.\neg A\ \mathit{true}$ gives us a witness t such that $[t/x]\neg A\ \mathit{true}$ is provable, and we can use t to refute $\forall x.A\ \mathit{true}$.

$$\dfrac{\dfrac{\dfrac{\overline{\exists x.\neg A\ \mathit{true}}^{\,w} \quad \dfrac{\overline{[a/x]\neg A\ \mathit{true}}^{\,y} \quad \dfrac{\overline{\forall x.A\ \mathit{true}}^{\,z}}{[a/x]A\ \mathit{true}}\forall\mathsf{E}}{\bot\ \mathit{true}}\neg\mathsf{E}}{\bot\ \mathit{true}}\exists\mathsf{E}^{a,y}}{\neg\forall x.A\ \mathit{true}}\neg\mathsf{I}^{z}}{\exists x.\neg A \supset \neg\forall x.A\ \mathit{true}}\supset\mathsf{I}^{w}$$

The converse $\neg\forall x.A \supset \exists x.\neg A\ \mathit{true}$ is not provable, however. Intuitively a proof of $\exists x.\neg A\ \mathit{true}$ requires a witness t such that $[t/x]\neg A\ \mathit{true}$ is provable, but no proof of $\neg\forall x.A\ \mathit{true}$ gives such a witness.

$$\dfrac{\dfrac{\dfrac{\overline{\neg\forall x.A\ \mathit{true}}^{\,w} \quad \begin{array}{c}?\\ \forall x.A\ \mathit{true}\end{array}}{\bot\ \mathit{true}}\neg\mathsf{E}}{\exists x.\neg A\ \mathit{true}}\bot\mathsf{E}}{\neg\forall x.A \supset \exists x.\neg A\ \mathit{true}}\supset\mathsf{I}^{w}$$

$$\frac{[a/x]A\ \mathit{true}}{\forall x.A\ \mathit{true}}\ \forall\mathsf{I}^{a} \qquad \frac{\forall x.A\ \mathit{true}}{[t/x]A\ \mathit{true}}\ \forall\mathsf{E} \qquad \frac{[t/x]A\ \mathit{true}}{\exists x.A\ \mathit{true}}\ \exists\mathsf{I} \qquad \frac{\exists x.A\ \mathit{true} \qquad \begin{array}{c}\overline{[a/x]A\ \mathit{true}}^{\,w}\\ \vdots\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\ \exists\mathsf{E}^{a,w}$$

Figure 1.1: Natural deduction system for first-order logic

Perhaps surprisingly, $(\forall x.A) \supset (\exists x.A)\ \mathit{true}$ is *not* provable. The reason is that although $\forall x.A\ \mathit{true}$ states that $[t/x]A\ \mathit{true}$ is provable for any term t, it does not decide a concrete term t such that $[t/x]A\ \mathit{true}$ is provable. In particular, if the set of terms is empty, $\forall x.A\ \mathit{true}$ holds trivially (because there is no term), but $\exists x.A\ \mathit{true}$ never holds because it is impossible to choose a term t for x, regardless of proposition A.

$$\cfrac{\cfrac{\cfrac{\overline{\forall x.A\ \mathit{true}}^{\,w}}{[t/x]A\ \mathit{true}?}\ \forall\mathsf{E}}{\exists x.A\ \mathit{true}}\ \exists\mathsf{I}}{(\forall x.A) \supset (\exists x.A)\ \mathit{true}}\ \supset\mathsf{I}^{w}$$

On the other hand, $\forall y.(\forall x.A) \supset (\exists x.A)\ \mathit{true}$ *is* provable even if y does not occur free in A. The difference from the previous example is that $\forall y$ allows us to make an assumption that the set of terms is not empty. In the proof shown below, parameter a denotes an arbitrary term in the set of terms, and its presence implies that the set of terms is not empty.

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{\forall x.A\ \mathit{true}}^{\,w}}{[a/x]A\ \mathit{true}}\ \forall\mathsf{E}}{\exists x.A\ \mathit{true}}\ \exists\mathsf{I}}{(\forall x.A) \supset (\exists x.A)\ \mathit{true}}\ \supset\mathsf{I}^{w}}{\forall y.(\forall x.A) \supset (\exists x.A)\ \mathit{true}}\ \forall\mathsf{I}^{a}$$

These two examples illustrate that in constructive logic, $\forall x.A$ is not equivalent to A even if x does not occur free in A at all: $\forall x.A$ asserts A on the assumption that the set of terms is not empty, whereas A without a universal quantifier cannot exploit such an assumption.

Figure 1.1 shows the inference rules for first-order logic.

Exercise 1.1. We have seen that a logical equivalence $\neg\forall x.A \ \equiv\ \exists x.\neg A$ fails. Check whether the following logical equivalence holds or not:

$$\neg\exists x.A \ \equiv\ \forall x.\neg A$$

Exercise 1.2. Suppose that term variable x is not free in proposition A, but free in proposition B. That is, we have $[t/x]A = A$, but $[t/x]B \neq B$ in general for an arbitrary term t. Check if each of the following judgments is provable.

- $(A \supset \forall x.B) \supset (\forall x.A \supset B)\ \mathit{true}$
- $(\forall x.A \supset B) \supset (A \supset \forall x.B)\ \mathit{true}$
- $(A \supset \exists x.B) \supset (\exists x.A \supset B)\ \mathit{true}$
- $(\exists x.A \supset B) \supset (A \supset \exists x.B)\ \mathit{true}$

1.5 Local soundness and completeness

To show the local soundness and completeness properties of first-order logic, we consider local reductions and expansions for universal and existential quantifications. A local reduction for universal

$$\frac{[a/x]A\!\uparrow}{\forall x.A\!\uparrow}\,\forall\mathsf{I}^a_\uparrow \qquad \frac{\forall x.A\!\downarrow}{[t/x]A\!\downarrow}\,\forall\mathsf{E}_\downarrow \qquad \frac{[t/x]A\!\uparrow}{\exists x.A\!\uparrow}\,\exists\mathsf{I}_\uparrow \qquad \frac{\exists x.A\!\downarrow \qquad \begin{array}{c}\overline{[a/x]A\!\downarrow}^{\,w}\\ \vdots\\ C\!\uparrow\end{array}}{C\!\uparrow}\,\exists\mathsf{E}^{a,w}_\uparrow$$

Figure 1.2: Natural deduction system for first-order logic

quantification is given as follows:

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ [a/x]A\ \mathit{true}\end{array}}{\forall x.A\ \mathit{true}}\,\forall\mathsf{I}^a}{[t/x]A\ \mathit{true}}\,\forall\mathsf{E} \qquad \Longrightarrow_R \qquad \begin{array}{c}[t/a]\mathcal{D}\\ [t/x]A\ \mathit{true}\end{array}$$

Since $\mathcal{D}$ proves $[a/x]A\ \mathit{true}$, we use $[t/a]\mathcal{D}$ to prove $[t/a][a/x]A\ \mathit{true} = [t/x]A\ \mathit{true}$. Similarly a local reduction for existential quantification shown below substitutes term t for parameter a:

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ [t/x]A\ \mathit{true}\end{array}}{\exists x.A\ \mathit{true}}\,\exists\mathsf{I} \qquad \left.\begin{array}{c}\overline{[a/x]A\ \mathit{true}}^{\,w}\\ \vdots\\ C\ \mathit{true}\end{array}\right\}\mathcal{E}}{C\ \mathit{true}}\,\exists\mathsf{E}^{a,w} \qquad \Longrightarrow_R \qquad \begin{array}{c}\mathcal{D}\\ \left.\begin{array}{c}[t/x]A\ \mathit{true}\\ \vdots\\ C\ \mathit{true}\end{array}\right\}[t/a]\mathcal{E}\end{array}$$

Note that $[t/a]\mathcal{E}$ proves the same judgment that $\mathcal{E}$ proves, namely $C\ \mathit{true}$, because parameter a does not appear in proposition C. On the other hand, $[t/a]\mathcal{E}$ changes $\overline{[a/x]A\ \mathit{true}}^{\,w}$ to $\overline{[t/a][a/x]A\ \mathit{true}}^{\,w}$, or $\overline{[t/x]A\ \mathit{true}}^{\,w}$, for which $\mathcal{D}$ is substituted. Local expansions for universal and existential quantifications are given as follows:

$$\begin{array}{c}\mathcal{E}\\ \forall x.A\ \mathit{true}\end{array} \quad \Longrightarrow_E \quad \cfrac{\cfrac{\begin{array}{c}\mathcal{E}\\ \forall x.A\ \mathit{true}\end{array}}{[a/x]A\ \mathit{true}}\,\forall\mathsf{E}}{\forall x.A\ \mathit{true}}\,\forall\mathsf{I}^a \qquad\qquad \begin{array}{c}\mathcal{E}\\ \exists x.A\ \mathit{true}\end{array} \quad \Longrightarrow_E \quad \cfrac{\begin{array}{c}\mathcal{E}\\ \exists x.A\ \mathit{true}\end{array} \qquad \cfrac{\overline{[a/x]A\ \mathit{true}}^{\,w}}{\exists x.A\ \mathit{true}}\,\exists\mathsf{I}}{\exists x.A\ \mathit{true}}\,\exists\mathsf{E}^{a,w}$$

Reading $\forall x.A$ as an infinite conjunction and $\exists x.A$ as an infinite disjunction gives the rules for deducing neutral and normal judgments in Figure 1.2. It turns out that Theorem **??** continues to hold in first-order logic, and proving $A\ \mathit{true}$ reduces to proving $A\!\uparrow$ as in propositional logic. Theorems **??** (normalization) and **??** (strong normalization) also continue to hold, provided that the following commuting conversion for existential quantification is available where the rule R is assumed to be an elimination rule:

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ \exists x.A\ \mathit{true}\end{array} \qquad \begin{array}{c}\overline{[a/x]A\ \mathit{true}}^{\,w}\\ \vdots\\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\,\exists\mathsf{E}^{a,w}}{C'\ \mathit{true}}\,R \qquad \Longrightarrow_C \qquad \cfrac{\begin{array}{c}\mathcal{D}\\ \exists x.A\ \mathit{true}\end{array} \qquad \begin{array}{c}\overline{[a/x]A\ \mathit{true}}^{\,w}\\ \vdots\\ \cfrac{C\ \mathit{true}}{C'\ \mathit{true}}\,R\end{array}}{C\ \mathit{true}}\,\exists\mathsf{E}^{a,w}$$

We can derive the above commuting conversion from the commuting conversion for $\vee$ by reading $\exists x.A$ as an infinite disjunction.

1.6 Examples

As a concrete example of reasoning in first-order logic, let us characterize natural numbers. We use $\mathbf{0}$ as a term denoting zero and s as a function symbol denoting the successor function. We also use three predicates: $\mathit{Nat}(t)$ to mean that t is a natural number, $\mathit{Eq}(t,t')$ to mean that t and t' are equal, and $\mathit{Lt}(t,t')$ to mean that t is less than t'.

First we need axioms as a means of defining the three predicates:

$$\dfrac{}{\mathit{Nat}(\mathbf{0})\ \mathit{true}}\ \mathit{Zero} \qquad \dfrac{}{\forall x.\mathit{Nat}(x) \supset \mathit{Nat}(\mathbf{s}(x))\ \mathit{true}}\ \mathit{Succ}$$

$$\dfrac{}{\forall x.\mathit{Eq}(x,x)\ \mathit{true}}\ \mathit{Eq}_i \qquad \dfrac{}{\forall x.\forall y.\forall z.(\mathit{Eq}(x,y) \wedge \mathit{Eq}(x,z)) \supset \mathit{Eq}(y,z)\ \mathit{true}}\ \mathit{Eq}_t$$

$$\dfrac{}{\forall x.\mathit{Lt}(x,\mathbf{s}(x))\ \mathit{true}}\ \mathit{Lt}_s \qquad \dfrac{}{\forall x.\forall y.\mathit{Eq}(x,y) \supset \neg \mathit{Lt}(x,y)\ \mathit{true}}\ \mathit{Lt}_\neg$$

The lower four axioms may be thought of as translations of the following mathematical properties:

- $x = x$.
- If $x = y$ and $x = z$, then $y = z$.
- $x < x + 1$.
- If $x = y$, then $x \not< y$.

Combined with these axioms, first-order logic allows us to prove new theorems about these predicates. As a trivial example, here is a proof of $\mathit{Nat}(\mathbf{s}(\mathbf{s}(\mathbf{0})))\ \mathit{true}$, which states that $\mathbf{s}(\mathbf{s}(\mathbf{0}))$ is a natural number:

$$\dfrac{\dfrac{\dfrac{}{\forall x.\mathit{Nat}(x) \supset \mathit{Nat}(\mathbf{s}(x))\ \mathit{true}}\ \mathit{Succ}}{\mathit{Nat}(\mathbf{s}(\mathbf{0})) \supset \mathit{Nat}(\mathbf{s}(\mathbf{s}(\mathbf{0})))\ \mathit{true}}\ \forall\mathsf{E} \quad \dfrac{\dfrac{\dfrac{}{\forall x.\mathit{Nat}(x) \supset \mathit{Nat}(\mathbf{s}(x))\ \mathit{true}}\ \mathit{Succ}}{\mathit{Nat}(\mathbf{0}) \supset \mathit{Nat}(\mathbf{s}(\mathbf{0}))\ \mathit{true}}\ \forall\mathsf{E} \quad \dfrac{}{\mathit{Nat}(\mathbf{0})\ \mathit{true}}\ \mathit{Zero}}{\mathit{Nat}(\mathbf{s}(\mathbf{0}))\ \mathit{true}}\ \supset\mathsf{E}}{\mathit{Nat}(\mathbf{s}(\mathbf{s}(\mathbf{0})))\ \mathit{true}}\ \supset\mathsf{E}$$

Note that the two applications of the rule $\supset$E substitute different terms, namely $\mathbf{s}(\mathbf{0})$ and $\mathbf{0}$, for term variable x in $\forall x.\mathit{Nat}(x) \supset \mathit{Nat}(\mathbf{s}(x))$.

An example of using an existential quantification is a proof of $\forall x.\mathit{Nat}(x) \supset (\exists y.\mathit{Nat}(y) \wedge \mathit{Eq}(x,y))\ \mathit{true}$ which states that if x is a natural number, there exists a natural number y such that $x = y$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathit{Nat}(a)\ \mathit{true}}\ z \quad \dfrac{\dfrac{}{\forall x.\mathit{Eq}(x,x)\ \mathit{true}}\ \mathit{Eq}_i}{\mathit{Eq}(a,a)\ \mathit{true}}\ \forall\mathsf{E}}{\mathit{Nat}(a) \wedge \mathit{Eq}(a,a)\ \mathit{true}}\ \wedge\mathsf{I}}{\exists y.\mathit{Nat}(y) \wedge \mathit{Eq}(a,y)\ \mathit{true}}\ \exists\mathsf{I}}{\mathit{Nat}(a) \supset (\exists y.\mathit{Nat}(y) \wedge \mathit{Eq}(a,y))\ \mathit{true}}\ \supset\mathsf{I}^z}{\forall x.\mathit{Nat}(x) \supset (\exists y.\mathit{Nat}(y) \wedge \mathit{Eq}(x,y))\ \mathit{true}}\ \forall\mathsf{I}^a$$

In the application of the rule $\exists$I, we use parameter a as a witness.

Here are two more examples. The first states the commutativity of equality: $x = y$ implies $y = x$ The second states that there is no term x such that $x = 0$ and $x = 1$.

- Proof of $\forall x.\forall y.\mathit{Eq}(x,y) \supset \mathit{Eq}(y,x)\ \mathit{true}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\forall y.\forall z.(\mathit{Eq}(x,y) \wedge \mathit{Eq}(x,z)) \supset \mathit{Eq}(y,z)\ \mathit{true}}\ \mathit{Eq}_t}{\forall y.\forall z.(\mathit{Eq}(a,y) \wedge \mathit{Eq}(a,z)) \supset \mathit{Eq}(y,z)\ \mathit{true}}\ \forall\mathsf{E}}{\forall z.(\mathit{Eq}(a,b) \wedge \mathit{Eq}(a,z)) \supset \mathit{Eq}(b,z)\ \mathit{true}}\ \forall\mathsf{E}}{(\mathit{Eq}(a,b) \wedge \mathit{Eq}(a,a)) \supset \mathit{Eq}(b,a)\ \mathit{true}}\ \forall\mathsf{E} \quad \dfrac{\dfrac{}{\mathit{Eq}(a,b)\ \mathit{true}}\ w \quad \dfrac{\dfrac{}{\forall x.\mathit{Eq}(x,x)\ \mathit{true}}\ \mathit{Eq}_i}{\mathit{Eq}(a,a)\ \mathit{true}}\ \forall\mathsf{E}}{\mathit{Eq}(a,b) \wedge \mathit{Eq}(a,a)\ \mathit{true}}\ \wedge\mathsf{I}}{\mathit{Eq}(b,a)\ \mathit{true}}\ \supset\mathsf{E}}{\mathit{Eq}(a,b) \supset \mathit{Eq}(b,a)\ \mathit{true}}\ \supset\mathsf{I}^w}{\forall y.\mathit{Eq}(a,y) \supset \mathit{Eq}(y,a)\ \mathit{true}}\ \forall\mathsf{I}^b}{\forall x.\forall y.\mathit{Eq}(x,y) \supset \mathit{Eq}(y,x)\ \mathit{true}}\ \forall\mathsf{I}^a$$

- Proof of $\neg\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathbf{s}(\mathbf{0}))\ \mathit{true}$:

$$
\dfrac{\dfrac{\dfrac{}{\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathbf{s}(\mathbf{0}))\ \mathit{true}}\,{}^{w} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\forall y.Eq(x,y) \supset \neg Lt(x,y)\ \mathit{true}}\,Lt_{\neg}}{\forall y.Eq(\mathbf{0},y) \supset \neg Lt(\mathbf{0},y)\ \mathit{true}}\,\forall\mathsf{E}}{Eq(\mathbf{0},\mathbf{s}(\mathbf{0})) \supset \neg Lt(\mathbf{0},\mathbf{s}(\mathbf{0}))\ \mathit{true}}\,\forall\mathsf{E} \qquad \begin{matrix}\mathcal{D}\\ Eq(\mathbf{0},\mathbf{s}(\mathbf{0}))\ \mathit{true}\end{matrix}}{\neg Lt(\mathbf{0},\mathbf{s}(\mathbf{0}))\ \mathit{true}}\,{\supset}\mathsf{E} \qquad \dfrac{\dfrac{}{\forall x.Lt(x,\mathbf{s}(x))\ \mathit{true}}\,Lt_{s}}{Lt(\mathbf{0},\mathbf{s}(\mathbf{0}))\ \mathit{true}}\,\forall\mathsf{E}}{\bot\ \mathit{true}}\,\neg\mathsf{E}}{\bot\ \mathit{true}}\,\exists\mathsf{E}^{a,z}}{\neg\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathbf{s}(\mathbf{0}))\ \mathit{true}}\,\neg\mathsf{I}^{w}
$$

where we let

$$
\mathcal{D} \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\forall y.\forall z.(Eq(x,y) \wedge Eq(x,z)) \supset Eq(y,z)\ \mathit{true}}\,Eq_{t}}{\forall y.\forall z.(Eq(a,y) \wedge Eq(a,z)) \supset Eq(y,z)\ \mathit{true}}\,\forall\mathsf{E}}{\forall z.(Eq(a,\mathbf{0}) \wedge Eq(a,z)) \supset Eq(\mathbf{0},z)\ \mathit{true}}\,\forall\mathsf{E}}{(Eq(a,\mathbf{0}) \wedge Eq(a,\mathbf{s}(\mathbf{0}))) \supset Eq(\mathbf{0},\mathbf{s}(\mathbf{0}))\ \mathit{true}}\,\forall\mathsf{E} \qquad \dfrac{}{Eq(a,\mathbf{0}) \wedge Eq(a,\mathbf{s}(\mathbf{0}))\ \mathit{true}}\,{}^{z}}{Eq(\mathbf{0},\mathbf{s}(\mathbf{0}))\ \mathit{true}}\,{\supset}\mathsf{E}
$$

1.7 Proof terms

As in propositional logic, we use the Curry-Howard isomorphism to represent proofs of truth judgments as proof terms. Proof terms for first-order logic are given as follows:

$$
\text{proof term} \quad M \quad ::= \quad \cdots \mid \lambda x.\, M \mid M\ t \mid \langle t, M\rangle \mid \mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ M
$$

$\lambda x.\, M$, a proof term of type $\forall x.A$, is a λ-abstraction that takes a term t and returns a proof term of type $[t/x]A$. (Recall that propositions and types are equivalent under the Curry-Howard isomorphism.) It is similar to a λ-abstraction from propositional logic except that it takes a term, instead of a proof term, as its argument. For example, given a term t (denoting a natural number), $\lambda x.\, M$ may return a proof term of type $Nat(t) \supset Nat(\mathbf{s}(t))$. A corresponding λ-application $M\ t$ is a proof term of type $[t/x]A$ if M is a proof term of type $\forall x.A$.

The typing rules for $\lambda x.\, M$ and $M\ t$ are given as follows:

$$
\dfrac{[a/x]M : [a/x]A}{\lambda x.\, M : \forall x.A}\,\forall\mathsf{I}^{a} \qquad \dfrac{M : \forall x.A}{M\ t : [t/x]A}\,\forall\mathsf{E}
$$

Here we write $[t/x]M$ for a substitution of term t for term variable x in proof term M. The rule $\forall\mathsf{I}^{a}$ proves that $\lambda x.\, M$ has type $\forall x.A$ by introducing a fresh parameter a and proving that $[a/x]M$ has type $[a/x]A$. For example, a proof that $\lambda x.\, M$ has type $\forall x.Eq(x,x)$ could show that $[a/x]M$ has type $Eq(a,a)$ for some parameter a. Note that in the rule $\forall\mathsf{I}^{a}$, term a appears in both proof term M and type A. This feature of first-order logic is manifested in the rule $\forall\mathsf{E}$: terms may appear not only in types but also in proof terms. Intuitively a proof about a specific term t needs to mention t somewhere in it. (Otherwise how can we prove a fact about t at all?) Hence a proof term whose type contains t also mentions t somewhere in it. For example, we could use $\mathsf{Eq_i}\ \mathbf{0}$ as a proof term of type $Eq(\mathbf{0},\mathbf{0})$ where $\mathsf{Eq_i}$ assumes type $\forall x.Eq(x,x)$. As a consequence, a substitution $[t/x]M$ on proof term M may need a substitution $[t/x]A$ on type A if x occurs inside A in M.

$\langle t, M\rangle$ is a proof term of type $\exists x.A$. Intuitively a proof of $\exists x.A\ \mathit{true}$ requires a concrete witness t and a proof that t satisfies A. Hence a proof term of type $\exists x.A$ contains such a witness t and a proof term M of type $[t/x]A$. For example, a proof term of type $\exists x.Eq(x,x)$ ("there exists a term x such that x is equal to x itself") may contain a witness $\mathbf{0}$ and a proof term of type $Eq(\mathbf{0},\mathbf{0})$ ("$\mathbf{0}$ is equal to $\mathbf{0}$). Thus we obtain the following typing rule for $\langle t, M\rangle$:

$$
\dfrac{M : [t/x]A}{\langle t, M\rangle : \exists x.A}\,\exists\mathsf{I}
$$

Given that M has type $\exists x.A$, a proof term $\mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ N$ decides the type of N after binding x and w to a witness t and a proof term of type $[t/x]A$, respectively. (Note that x is a term variable

$$\frac{[a/x]M : [a/x]A}{\lambda x.\, M : \forall x.A}\,\forall\mathsf{I}^a \qquad \frac{M : \forall x.A}{M\,t : [t/x]A}\,\forall\mathsf{E} \qquad \frac{M : [t/x]A}{\langle t, M\rangle : \exists x.A}\,\exists\mathsf{I} \qquad \frac{M : \exists x.A \quad \begin{array}[b]{c}\overline{w : [a/x]A}\\ \vdots \\ [a/x]N : C\end{array}}{\mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ N : C}\,\exists\mathsf{E}^a$$

Figure 1.3: Typing rules for proof terms in first-order logic

whereas w is a variable ranging over proof terms.) Since such a witness is unknown in general (*e.g.*, if M is a variable), we have to assume an arbitrary witness a and assign type $[a/x]A$ to w. Accordingly we replace x in N by a. Thus we obtain the following typing rule for $\mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ N$:

$$\frac{M : \exists x.A \quad \begin{array}[b]{c}\overline{w : [a/x]A}\\ \vdots \\ [a/x]N : C\end{array}}{\mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ N : C}\,\exists\mathsf{E}^a$$

In practice, we may use the following typing rule with an extra assumption that x is a fresh term variable:

$$\frac{M : \exists x.A \quad \begin{array}[b]{c}\overline{w : A}\\ \vdots \\ N : C\end{array}}{\mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ N : C}\,\exists\mathsf{E}$$

Figure 1.3 shows all the typing rules for proof terms in first-order logic.

Exercise 1.3. Rewrite these typing rules using hypothetical judgments.

We derive β-reductions and η-expansions for universal and existential quantifications from their corresponding local reductions and expansions of proofs given in Section 1.5. For universal quantifications, we assign proof terms as follows:

$$\frac{\dfrac{[a/x]M : [a/x]A}{\lambda x.\, M : \forall x.A}\,\forall\mathsf{I}^a}{(\lambda x.\, M)\,t : [t/x]A}\,\forall\mathsf{E} \quad \Longrightarrow_\beta \quad [t/a][a/x]M : [t/a][a/x]A$$

$$M : \forall x.A \quad \Longrightarrow_\eta \quad \frac{\dfrac{M : \forall x.A}{M\ a : [a/x]A}\,\forall\mathsf{E}}{\lambda x.\, M\ x : \forall x.A}\,\forall\mathsf{I}^a \quad \textit{where } M\ a = [a/x](M\ x)$$

As we have $[t/a][a/x]M = [t/a]M$ (and $[t/a][a/x]A = [t/x]A$), we obtain the following β-reduction and η-expansion:

$$\begin{array}{lll} (\lambda x.\, M)\ t & \Longrightarrow_\beta & [t/x]M \\ M : \forall x.A & \Longrightarrow_\eta & \lambda x.\, M\ x \quad (x \textit{ is not free in } M) \end{array}$$

For existential quantifications, we assign proof terms as follows:

$$\frac{\dfrac{M : [t/x]A}{\langle t, M\rangle : \exists x.A}\,\exists\mathsf{I} \quad \begin{array}[b]{c}\overline{w : [a/x]A}\\ \vdots \\ [a/x]N : C\end{array}}{\mathsf{let}\ \langle x, w\rangle = \langle t, M\rangle\ \mathsf{in}\ N : C}\,\exists\mathsf{E}^{a,w} \quad \Longrightarrow_\beta \quad \begin{array}{c}[M/w][t/a]w : [t/a][a/x]A \\ \vdots \\ [M/w][t/a][a/x]N : C\end{array}$$

$$M : \exists x.A \quad \Longrightarrow_\eta \quad \frac{M : \exists x.A \quad \dfrac{\overline{w : [a/x]A}}{\langle a, w\rangle : \exists x.A}\,\exists\mathsf{I}}{\mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ \langle x, w\rangle : \exists x.A}\,\exists\mathsf{E}^a \quad \textit{where } \langle a, w\rangle = [a/x]\langle x, w\rangle$$

As we have $[M/w][t/a][a/x]N = [M/w][t/x]N$, we obtain the following β-reduction and η-expansion:

$$\begin{array}{rcl} \mathsf{let}\ \langle x, w\rangle = \langle t, M\rangle\ \mathsf{in}\ N & \Longrightarrow_\beta & [M/w][t/x]N \\ M : \exists x.A & \Longrightarrow_\eta & \mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ \langle x, w\rangle \end{array}$$

As in Section **??**, we extend the definition of elim terms and intro terms by using an elim term E to represent a proof of $A{\downarrow}$ and an intro term I to represent a proof of $A{\uparrow}$. According to the rules for deducing neutral and normal judgments given in Section 1.5, we obtain the following definition:

$$\begin{array}{lrcl} \text{elim term} & E & ::= & \cdots \mid E\ t \\ \text{intro term} & I & ::= & \cdots \mid \lambda x.\, I \mid \langle t, I\rangle \mid \mathsf{let}\ \langle x, w\rangle = E\ \mathsf{in}\ I \end{array}$$

The commuting conversion for existential quantification requires us to extend the definition of commuting conversion contexts explained in Section **??** and introduce another case for $M \Longrightarrow_c N$ as shown below:

$$\begin{array}{rcl} \text{commuting conversion context} \quad \kappa & ::= & \cdots \mid \Box\ t \mid \mathsf{let}\ \langle x, w\rangle = \Box\ \mathsf{in}\ I \\ \kappa[\![\mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ N]\!] & \Longrightarrow_c & \mathsf{let}\ \langle x, w\rangle = M\ \mathsf{in}\ \kappa[\![N]\!] \end{array}$$

1.8 Examples of proof terms

This section rewrites all the proofs in Section 1.6 using proof terms. First we need constant proof terms for axioms:

$$\dfrac{}{\mathsf{Nat_0} : Nat(\mathbf{0})}\ Zero \qquad \dfrac{}{\mathsf{Nat_s} : \forall x.Nat(x) \supset Nat(\mathbf{s}(x))}\ Succ$$

$$\dfrac{}{\mathsf{Eq_i} : \forall x.Eq(x, x)}\ Eq_i \qquad \dfrac{}{\mathsf{Eq_t} : \forall x.\forall y.\forall z.(Eq(x, y) \wedge Eq(x, z)) \supset Eq(y, z)}\ Eq_t$$

$$\dfrac{}{\mathsf{Lt_s} : \forall x.Lt(x, \mathbf{s}(x))}\ Lt_s \qquad \dfrac{}{\mathsf{Lt_\neg} : \forall x.\forall y.Eq(x, y) \supset \neg Lt(x, y)}\ Lt_\neg$$

The proof of $Nat(\mathbf{s}(\mathbf{s}(\mathbf{0})))$ *true* corresponds to a proof term $\mathsf{Nat_s}\ \mathbf{s}(\mathbf{0})\ (\mathsf{Nat_s}\ \mathbf{0}\ \mathsf{Nat_0})$ as shown in the following derivation tree:

$$\dfrac{\dfrac{\dfrac{}{\mathsf{Nat_s} : \forall x.Nat(x) \supset Nat(\mathbf{s}(x))}\ Succ}{\mathsf{Nat_s}\ \mathbf{s}(\mathbf{0}) : Nat(\mathbf{s}(\mathbf{0})) \supset Nat(\mathbf{s}(\mathbf{s}(\mathbf{0})))}\ \forall\mathsf{E} \qquad \dfrac{\dfrac{\dfrac{}{\mathsf{Nat_s} : \forall x.Nat(x) \supset Nat(\mathbf{s}(x))}\ Succ}{\mathsf{Nat_s}\ \mathbf{0} : Nat(\mathbf{0}) \supset Nat(\mathbf{s}(\mathbf{0}))}\ \forall\mathsf{E} \qquad \dfrac{}{\mathsf{Nat_0} : Nat(\mathbf{0})}\ Zero}{\mathsf{Nat_s}\ \mathbf{0}\ \mathsf{Nat_0} : Nat(\mathbf{s}(\mathbf{0}))}\ {\supset}\mathsf{E}}{\mathsf{Nat_s}\ \mathbf{s}(\mathbf{0})\ (\mathsf{Nat_s}\ \mathbf{0}\ \mathsf{Nat_0}) : Nat(\mathbf{s}(\mathbf{s}(\mathbf{0})))}\ {\supset}\mathsf{E}$$

In the same fashion, we obtain the following proof terms:

- Proof term of type $\forall x.Nat(x) \supset (\exists y.Nat(y) \wedge Eq(x, y))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{z : Nat(a)} \qquad \dfrac{\dfrac{}{\mathsf{Eq_i} : \forall x.Eq(x, x)}\ Eq_i}{\mathsf{Eq_i}\ a : Eq(a, a)}\ \forall\mathsf{E}}{(z, \mathsf{Eq_i}\ a) : Nat(a) \wedge Eq(a, a)}\ \wedge\mathsf{I}}{\langle a, (z, \mathsf{Eq_i}\ a)\rangle : \exists y.Nat(y) \wedge Eq(a, y)}\ \exists\mathsf{I}}{\lambda z\!:\!Nat(a).\ \langle a, (z, \mathsf{Eq_i}\ a)\rangle : Nat(a) \supset (\exists y.Nat(y) \wedge Eq(a, y))}\ {\supset}\mathsf{I}^z}{\lambda x.\ \lambda z\!:\!Nat(x).\ \langle x, (z, \mathsf{Eq_i}\ x)\rangle : \forall x.Nat(x) \supset (\exists y.Nat(y) \wedge Eq(x, y))}\ \forall\mathsf{I}^a$$

- Proof term of type $\forall x.\forall y.Eq(x, y) \supset Eq(y, x)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{Eq_t} : \forall x.\forall y.\forall z.(Eq(x, y) \wedge Eq(x, z)) \supset Eq(y, z)}\ Eq_t}{\mathsf{Eq_t}\ a : \forall y.\forall z.(Eq(a, y) \wedge Eq(a, z)) \supset Eq(y, z)}\ \forall\mathsf{E}}{\mathsf{Eq_t}\ a\ b : \forall z.(Eq(a, b) \wedge Eq(a, z)) \supset Eq(b, z)}\ \forall\mathsf{E}}{\mathsf{Eq_t}\ a\ b\ a : (Eq(a, b) \wedge Eq(a, a)) \supset Eq(b, a)}\ \forall\mathsf{E} \qquad \dfrac{\dfrac{}{w : Eq(a, b)} \qquad \dfrac{\dfrac{}{\mathsf{Eq_i} : \forall x.Eq(x, x)}\ Eq_i}{\mathsf{Eq_i}\ a : Eq(a, a)}\ \forall\mathsf{E}}{(w, \mathsf{Eq_i}\ a) : Eq(a, b) \wedge Eq(a, a)}\ \wedge\mathsf{I}}{\mathsf{Eq_t}\ a\ b\ a\ (w, \mathsf{Eq_i}\ a) : Eq(b, a)}\ {\supset}\mathsf{E}}{\lambda w\!:\!Eq(a, b).\ \mathsf{Eq_t}\ a\ b\ a\ (w, \mathsf{Eq_i}\ a) : Eq(a, b) \supset Eq(b, a)}\ {\supset}\mathsf{I}^w}{\lambda y.\ \lambda w\!:\!Eq(a, y).\ \mathsf{Eq_t}\ a\ y\ a\ (w, \mathsf{Eq_i}\ a) : \forall y.Eq(a, y) \supset Eq(y, a)}\ \forall\mathsf{I}^b}{\lambda x.\ \lambda y.\ \lambda w\!:\!Eq(x, y).\ \mathsf{Eq_t}\ x\ y\ x\ (w, \mathsf{Eq_i}\ x) : \forall x.\forall y.Eq(x, y) \supset Eq(y, x)}\ \forall\mathsf{I}^a$$

- Proof term of type $\neg\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathrm{s}(\mathbf{0}))$:

$$\dfrac{\dfrac{\dfrac{}{w : \exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathrm{s}(\mathbf{0}))} \qquad \dfrac{\overset{\mathcal{E}}{(\mathsf{Lt}_\neg\ \mathbf{0}\ \mathrm{s}(\mathbf{0}))\ (\mathsf{Eq_t}\ a\ \mathbf{0}\ \mathrm{s}(\mathbf{0})\ z) : \neg Lt(\mathbf{0},\mathrm{s}(\mathbf{0}))} \quad \dfrac{\dfrac{}{\mathsf{Lt_s} : \forall x.Lt(x,\mathrm{s}(x))}\,Lt_s}{\mathsf{Lt_s}\ \mathbf{0} : Lt(\mathbf{0},\mathrm{s}(\mathbf{0}))}\,\forall\mathsf{E}}{(\mathsf{Lt}_\neg\ \mathbf{0}\ \mathrm{s}(\mathbf{0}))\ (\mathsf{Eq_t}\ a\ \mathbf{0}\ \mathrm{s}(\mathbf{0})\ z)\ (\mathsf{Lt_s}\ \mathbf{0}) : \bot}\,\neg\mathsf{E}}{\mathsf{let}\ \langle x,z\rangle = w\ \mathsf{in}\ (\mathsf{Lt}_\neg\ \mathbf{0}\ \mathrm{s}(\mathbf{0}))\ (\mathsf{Eq_t}\ x\ \mathbf{0}\ \mathrm{s}(\mathbf{0})\ z)\ (\mathsf{Lt_s}\ \mathbf{0}) : \bot}\,\exists\mathsf{E}^a}{\lambda w{:}\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathrm{s}(\mathbf{0})).\ \mathsf{let}\ \langle x,z\rangle = w\ \mathsf{in}\ (\mathsf{Lt}_\neg\ \mathbf{0}\ \mathrm{s}(\mathbf{0}))\ (\mathsf{Eq_t}\ x\ \mathbf{0}\ \mathrm{s}(\mathbf{0})\ z)\ (\mathsf{Lt_s}\ \mathbf{0}) : \neg\exists x.Eq(x,\mathbf{0}) \wedge Eq(x,\mathrm{s}(\mathbf{0}))}\,\neg\mathsf{I}^w$$

where we let

$$\mathcal{E} \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{Lt}_\neg : \forall x.\forall y.Eq(x,y) \supset \neg Lt(x,y)}\,Lt_\neg}{\mathsf{Lt}_\neg\ \mathbf{0} : \forall y.Eq(\mathbf{0},y) \supset \neg Lt(\mathbf{0},y)}\,\forall\mathsf{E}}{\mathsf{Lt}_\neg\ \mathbf{0}\ \mathrm{s}(\mathbf{0}) : Eq(\mathbf{0},\mathrm{s}(\mathbf{0})) \supset \neg Lt(\mathbf{0},\mathrm{s}(\mathbf{0}))}\,\forall\mathsf{E} \qquad \overset{\mathcal{D}}{\mathsf{Eq_t}\ a\ \mathbf{0}\ \mathrm{s}(\mathbf{0})\ z : Eq(\mathbf{0},\mathrm{s}(\mathbf{0}))}}{(\mathsf{Lt}_\neg\ \mathbf{0}\ \mathrm{s}(\mathbf{0}))\ (\mathsf{Eq_t}\ a\ \mathbf{0}\ \mathrm{s}(\mathbf{0})\ z) : \neg Lt(\mathbf{0},\mathrm{s}(\mathbf{0}))}\,\supset\mathsf{E}$$

where we let

$$\mathcal{D} \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{Eq_t} : \forall x.\forall y.\forall z.(Eq(x,y) \wedge Eq(x,z)) \supset Eq(y,z)}\,Eq_t}{\mathsf{Eq_t}\ a : \forall y.\forall z.(Eq(a,y) \wedge Eq(a,z)) \supset Eq(y,z)}\,\forall\mathsf{E}}{\mathsf{Eq_t}\ a\ \mathbf{0} : \forall z.(Eq(a,\mathbf{0}) \wedge Eq(a,z)) \supset Eq(\mathbf{0},z)}\,\forall\mathsf{E}}{\mathsf{Eq_t}\ a\ \mathbf{0}\ \mathrm{s}(\mathbf{0}) : (Eq(a,\mathbf{0}) \wedge Eq(a,\mathrm{s}(\mathbf{0}))) \supset Eq(\mathbf{0},\mathrm{s}(\mathbf{0}))}\,\forall\mathsf{E} \qquad \dfrac{}{z : Eq(a,\mathbf{0}) \wedge Eq(a,\mathrm{s}(\mathbf{0}))}}{\mathsf{Eq_t}\ a\ \mathbf{0}\ \mathrm{s}(\mathbf{0})\ z : Eq(\mathbf{0},\mathrm{s}(\mathbf{0}))}\,\supset\mathsf{E}$$