A Modal Logic Internalizing Normal Proofs

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Abstract

In the proof-theoretic study of logic, the notion of normal proof has been understood and investigated as a metalogical property. Usually we formulate a system of logic, identify a class of proofs as normal proofs, and show that every proof in the system reduces to a corresponding normal proof. This paper develops a system of modal logic that is capable of expressing the notion of normal proof *within the system itself*, thereby making normal proofs an inherent property of the logic. Using a modality $\triangle$ to express the existence of a normal proof, the system provides a means for both recognizing and manipulating its own normal proofs. We develop the system as a sequent calculus with the implication connective $\supset$ and the modality $\triangle$, and prove the cut elimination theorem. From the sequent calculus, we derive two equivalent natural deduction systems.

Keywords: Normal proof, Modal logic, Sequent calculus, Natural deduction system, Reflective system

1. Introduction

In the proof-theoretic study of logic, the notion of normal proof has played a central role. Conceptually a normal proof is a proof that contains no indirect reasoning and provides direct evidence of the truth it asserts. In a formal system of logic, a normal proof is a proof that contains no "detour" which introduces a connective or modality (*e.g.*, implication $\supset$, conjunction $\wedge$, and necessity $\Box$) only to immediately eliminate it. In this sense, a normal proof is minimal in size because it does not reduce to another proof (irrespective of the size of its representation).

Traditionally the notion of normal proof has been understood and investigated as a metalogical property. Usually we formulate a system of logic, identify a class of proofs as normal proofs, and show that every proof in the system reduces to a corresponding (unique) normal proof. The result is then exploited in examining other metalogical properties of the system such as unprovability of inconsistency ($\perp$) and solving practical problems such as building a theorem prover. Thus, although not recognized by the system itself, normal proofs serve as an indispensable tool for succinctly characterizing the system and developing practical applications.

This paper develops a system of logic that is capable of expressing the notion of normal proof *within the system itself*. That is, the system has a means for internalizing and reasoning about its

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own normal proofs, thereby making normal proofs an inherent property of the logic (as opposed to a metalogical property). Thus the system is reflective [1] in that it is self-aware of its own normal proofs. To the best of our knowledge, no such system has been proposed.

We formulate the logic in the judgmental style of Martin-Löf [2, 3] which distinguishes between *judgments* and *propositions*. A judgment represents an object of knowledge and a proof of it allows us to know the object of knowledge. If we do not have a proof, the judgment is not part of our knowledge. In contrast, a proposition conveys no knowledge in itself, but if A is known to be a proposition, we know what counts as a verification of its truth. That is, we can check whether a proof of the truth of A is indeed valid or not. Thus the notion of judgment is independent of (and precedes in priority) the notion of proposition.

In order to deal with both ordinary proofs and normal proofs within the same system of logic, we use two separate judgments $A\ \mathit{true}$ and $A\!\uparrow$ where A is a proposition, or simply a formula. $A\ \mathit{true}$ is a *truth judgment* whose proof is an ordinary proof and may reduce to another proof. $A\!\uparrow$, adopted from the intercalation calculus of Byrnes [4], is a *normality judgment* whose proof is a normal proof and does not reduce to another proof. As $A\!\uparrow$ states a different "mode" of truth, namely truth with a normal proof, we develop the system as "modal" logic [5] by defining a new modality $\triangle$ to capture the metalogical property of $A\ \mathit{true}$ expressed in $A\!\uparrow$. As shown by Pfenning and Davies [3], the judgmental style lends itself particularly well to the development of systems of modal logic. In our case, we use $\triangle$ to internalize a normality judgment within a truth judgment:[1]

$$\frac{A\!\uparrow}{\triangle A\ \mathit{true}}\ \triangle\mathsf{I}$$

A technical challenge is to deal with the chicken-and-egg nature of the problem. The rule $\triangle\mathsf{I}$ internalizes $A\!\uparrow$ within $\triangle A\ \mathit{true}$ and thus expands the set of truth judgments. At the same time, it also expands the set of normality judgments because the existence of $\triangle A\ \mathit{true}$ implies the existence of a corresponding normality judgment $\triangle A\!\uparrow$. Now the rule $\triangle\mathsf{I}$ allows us to deduce another truth judgment $\triangle\triangle A\ \mathit{true}$, which introduces yet another normality judgment $\triangle\triangle A\!\uparrow$, and so on. As is the case in similar reflective systems [6, 7, 1], the problem is quite subtle, especially because we wish to develop a simple system using a single modality $\triangle$ instead of an infinite tower of modalities (*e.g.*, $\frac{A\uparrow_1}{\triangle_1 A\ \mathit{true}}$, $\frac{A\uparrow_2}{\triangle_2 A\uparrow_1}$, $\frac{A\uparrow_3}{\triangle_3 A\uparrow_2}$, $\cdots$).

We develop a sequent calculus with the implication connective $\supset$ and the modality $\triangle$, and prove the cut elimination theorem. Then we derive two natural deduction system equivalent to the sequent calculus (one for deducing normality judgments and another for deducing truth judgments). Our finding is that in order for the system to be useful and interesting, $\triangle$ should be used to internalize not a normality judgment in the standard sense, but a weaker form of normality judgment whose proof may use hypotheses of normality judgments.

From a philosophical point of view, our system is superficially similar to provability logic [8] in that it is concerned with provability of judgments and is also reflective. Its real nature is different, however, because the modality $\triangle$ expresses not the general notion of provability but

[1] To internalize a judgment J means to represent the knowledge expressed in J with a truth judgment using a specific connective or modality. For example, we internalize a hypothetical judgment $\begin{array}{c}\overline{A\ \mathit{true}}\\ \vdots \\ B\ \mathit{true}\end{array}$ within a truth judgment $A \supset B\ \mathit{true}$ using the implication connective $\supset$. In modal logic, we internalize within a truth judgment $\Box A\ \mathit{true}$ a truth judgment $A\ \mathit{true}$ that is valid in every context.

only the existence of a special form of proof, namely a weaker form of normality judgment. For example, $\triangle(\triangle A \supset A)$ *true* is provable in our system, but not in provability logic if $\triangle$ is used as the provability modality. (In provability logic, $(\triangle(\triangle A \supset A)) \supset \triangle A$ *true* is given as an axiom.) $\triangle$ is also different from the necessity modality $\Box$ in modal logic: $\Box$ is concerned with proofs valid in every context, which are not necessarily normal. For example, $\triangle(A \supset B) \supset (\triangle A \supset \triangle B)$ *true* is not provable in our system whereas $\Box(A \supset B) \supset (\Box A \supset \Box B)$ *true* is provable in modal logic like S4.

While it is interesting mainly from a theoretical point of view, our system has indeed been inspired by a practical type system for parallel functional languages [9]. The type system uses a modality $\boxdot$ to indicate whether the result of evaluating a given term contains mutable references or not. As a term always evaluates to a value, the type system is inherently capable of recognizing values, which are a special class of irreducible terms. By applying the Curry-Howard isomorphism, we obtain a system of logic recognizing a special class of proofs that can be represented by values. Our system attempts to further generalize the correspondence by using the modality $\triangle$ to recognize fully normal proofs. Thus it may serve as a proof-theoretic foundation for type systems that distinguish between different classes of terms belonging to the same type.

We begin in Section 2 by introducing the problem of internalizing normality judgments with the modality $\triangle$.

2. Modality $\triangle$ for internalizing normal proofs

This section specifies our goal and explains technical difficulties in detail. Because of the peculiarity of internalizing normality judgments with the new modality $\triangle$, we need to introduce some new concepts not found in the conventional proof-theoretic study of logic. We consider a fragment of propositional logic with the implication connective $\supset$ only, since all interesting challenges arise from the interaction between $\triangle$ and $\supset$.

2.1. Goal

Consider the following natural deduction system $\mathsf{N}^{\mathsf{true}}$:

$$\left.\begin{array}{cc} \dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x} \\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}} \supset\mathsf{I}^x & \qquad \dfrac{A \supset B\ \mathit{true} \quad A\ \mathit{true}}{B\ \mathit{true}} \supset\mathsf{E} \end{array}\right\}\mathsf{N}^{\mathsf{true}}$$

A proof of a truth judgment *A true* is normal if it contains no "detour" in which an introduction of $\supset$ (by the introduction rule $\supset$I) is immediately followed by its elimination (by the elimination rule $\supset$E).[2] We can always remove such a detour by replacing a hypothesis with another existing

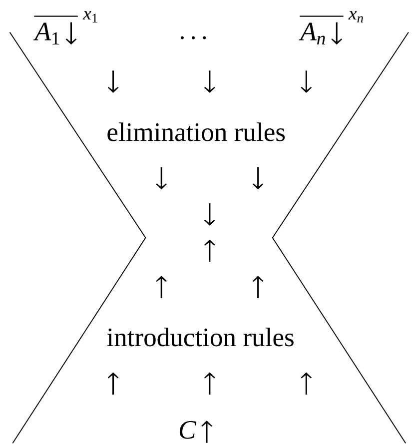
[2]For the fragment of propositional logic with $\supset$ only, this definition of normal proof is accurate. For full propositional logic, this definition needs to be extended.

proof:[3]

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,x}\\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\ \supset\!\mathsf{I}^x \quad \begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array}}{B\ \mathit{true}}\ \supset\!\mathsf{E} \qquad \longrightarrow \qquad \begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\\ \vdots \\ B\ \mathit{true}\end{array}$$

Prawitz [10] refers to the sequence of formulae in a detour as a maximal segment and defines a normal proof as a proof with no maximal segment. Instead of characterizing normal proofs in terms of absence of detours or maximal segments, we adopt the intercalation calculus of Byrnes [4] which can be thought of as using a new form of judgment to directly define normal proofs.

In addition to truth judgments, our system uses two new basic judgments: *normality judgments* and *neutrality judgments*. We use a normality judgment $A\!\uparrow$ for a normal proof of A *true*. The intuition is that a proof of $A\!\uparrow$ requires a bottom-up application of an introduction rule (such as $\supset$I). We use a neutrality judgment $A\!\downarrow$ for a *neutral proof* of A *true*. A neutral proof is either a hypothesis or obtained as a top-down application of an elimination rule (such as $\supset$E) to another neutral proof. When $A\!\downarrow$ and $A\!\uparrow$ meet in the middle, we complete a normal proof. Thus a proof of a normality judgment $C\!\uparrow$ has the following structure (which clearly shows that a normal proof contains no detour):



The following natural deduction system $\mathsf{N}^{\uparrow\!\downarrow}$ gives the inference rules for normality and neutrality judgments:

$$\left.\begin{array}{ccc}\dfrac{\begin{array}{c}\overline{A\!\downarrow}^{\,u}\\ \vdots\\ B\!\uparrow\end{array}}{A \supset B\!\uparrow}\ \supset\!\mathsf{I}^u_{\uparrow} & \dfrac{A \supset B\!\downarrow \quad A\!\uparrow}{B\!\downarrow}\ \supset\!\mathsf{E}_{\downarrow} & \dfrac{A\!\downarrow}{A\!\uparrow}\ \downarrow\!\uparrow\end{array}\right\}\ \mathsf{N}^{\uparrow\!\downarrow}$$

Note that $A\!\downarrow$ is strictly stronger than $A\!\uparrow$ because of the rule $\downarrow\!\uparrow$.

[3] We write $\begin{array}{c}\mathcal{D}\\ A\ \mathit{true}\end{array}$ for a proof $\mathcal{D}$ of A *true*.

The rules in $\mathsf{N}^{\uparrow\downarrow}$ may be thought of as specifying a special strategy in the search of proofs of truth judgments in $\mathsf{N}^{\mathsf{true}}$, since a proof of $A\uparrow$ can be converted to a proof of $A\ \mathit{true}$ by replacing all normality and neutrality judgments in it with truth judgments. Hence $A\uparrow$ is stronger than $A\ \mathit{true}$. The normalization theorem [10], however, states that $\mathsf{N}^{\mathsf{true}}$ and $\mathsf{N}^{\uparrow\downarrow}$ are in fact equivalent: $A\ \mathit{true}$ is provable in $\mathsf{N}^{\mathsf{true}}$ if and only if $A\uparrow$ is provable in $\mathsf{N}^{\uparrow\downarrow}$.

Proposition 2.1 ($\mathsf{N}^{\mathsf{true}} = \mathsf{N}^{\uparrow\downarrow}$)**.** *$A$ true in $\mathsf{N}^{\mathsf{true}}$ if and only if $A\uparrow$ in $\mathsf{N}^{\uparrow\downarrow}$.*

In order to define the modality $\triangle$ capturing the metalogical property of $A\ \mathit{true}$ expressed in $A\uparrow$, we combine $\mathsf{N}^{\mathsf{true}}$ and $\mathsf{N}^{\uparrow\downarrow}$ via the following rules:

$$\frac{A\uparrow}{\triangle A\ \mathit{true}}\ \triangle\mathsf{I} \qquad\qquad \frac{\triangle A\ \mathit{true} \qquad \begin{array}{c}\overline{A\uparrow}^{\,v}\\ \vdots \\ B\ \mathit{true}\end{array}}{B\ \mathit{true}}\ \triangle\mathsf{E}^{v}$$

The rule $\triangle\mathsf{I}$ internalizes a normality judgment $A\uparrow$ within a truth judgment using the modality $\triangle$. The rule $\triangle\mathsf{E}$ enables us to extract the normality judgment $A\uparrow$ internalized into a truth judgment $\triangle A\ \mathit{true}$. Note that like a hypothesis $\overline{A\ \mathit{true}}$ of a truth judgment, a hypothesis $\overline{A\uparrow}$ of a normality judgment can be interpreted literally. That is, it just assumes a proof of a normality judgment $A\uparrow$.

The two rules $\triangle\mathsf{I}$ and $\triangle\mathsf{E}$ satisfy local soundness and completeness in the following sense [3]:

- An introduction followed by an elimination can be reduced.

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}\\ A\uparrow\end{array}}{\triangle A\ \mathit{true}}\ \triangle\mathsf{I} \qquad \begin{array}{c}\overline{A\uparrow}^{\,v}\\ \vdots \\ B\ \mathit{true}\end{array}}{B\ \mathit{true}}\ \triangle\mathsf{E}^{v} \quad \begin{array}{c}\text{reduction}\\ \longrightarrow\end{array} \quad \begin{array}{c}\mathcal{D}\\ A\uparrow\\ \vdots\\ B\ \mathit{true}\end{array}$$

- A proof of $\triangle A\ \mathit{true}$ can be expanded into another proof of $\triangle A\ \mathit{true}$ via an elimination by the rule $\triangle\mathsf{E}$.

$$\begin{array}{c}\mathcal{E}\\ \triangle A\ \mathit{true}\end{array} \quad \begin{array}{c}\text{expansion}\\ \longrightarrow\end{array} \quad \frac{\begin{array}{c}\mathcal{E}\\ \triangle A\ \mathit{true}\end{array} \qquad \dfrac{\overline{A\uparrow}^{\,v}}{\triangle A\ \mathit{true}}\ \triangle\mathsf{I}}{\triangle A\ \mathit{true}}\ \triangle\mathsf{E}^{v}$$

Now that we have inference rules for truth judgments $\triangle A\ \mathit{true}$, we also need corresponding inference rules for normality and neutrality judgments:

$$\frac{A\uparrow}{\triangle A\uparrow}\ \triangle\mathsf{I}_{\uparrow} \qquad\qquad \frac{\triangle A\downarrow \qquad \begin{array}{c}\overline{A\uparrow}^{\,w}\\ \vdots \\ B\uparrow\end{array}}{B\uparrow}\ \triangle\mathsf{E}^{w}_{\downarrow}$$

The rule $\triangle\mathsf{I}_{\uparrow}$, derived from the rule $\triangle\mathsf{I}$, explains how to build a new proof of $\triangle A\uparrow$. The rule $\triangle\mathsf{E}_{\downarrow}$, derived from the rule $\triangle\mathsf{E}$, explains how to exploit an existing proof of $\triangle A\downarrow$. Without these rules, the system does not fully capture the notion of normal proof because not every formula A is allowed in $\triangle A$ *true* (*e.g.*, $A = \triangle A'$).

We write $\mathsf{N}^{\mathsf{true}}_{\triangle}$ for the natural deduction system consisting of all the inference rules given above; we also write $\mathsf{N}^{\uparrow\downarrow}_{\triangle}$ for the natural deduction system consisting of $\mathsf{N}^{\uparrow\downarrow}$, $\triangle\mathsf{I}_{\uparrow}$, and $\triangle\mathsf{E}_{\downarrow}$, which deals only with normality and neutrality judgments:

$$\begin{array}{lcl} \mathsf{N}^{\mathsf{true}}_{\triangle} & = & \mathsf{N}^{\mathsf{true}} + \triangle\mathsf{I}, \triangle\mathsf{E} + \mathsf{N}^{\uparrow\downarrow}_{\triangle} \\ \mathsf{N}^{\uparrow\downarrow}_{\triangle} & = & \mathsf{N}^{\uparrow\downarrow} + \triangle\mathsf{I}_{\uparrow}, \triangle\mathsf{E}_{\downarrow} \end{array}$$

Our goal is to make $\mathsf{N}^{\mathsf{true}}_{\triangle}$ equivalent to $\mathsf{N}^{\uparrow\downarrow}_{\triangle}$, revising both systems as necessary, in the same way that $\mathsf{N}^{\mathsf{true}}$ is equivalent to $\mathsf{N}^{\uparrow\downarrow}$:

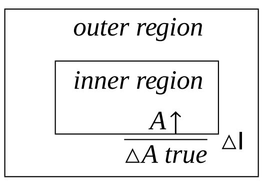
Conjecture 2.2 ($\mathsf{N}^{\mathsf{true}}_{\triangle} = \mathsf{N}^{\uparrow\downarrow}_{\triangle}$)**.** *$A$ true in $\mathsf{N}^{\mathsf{true}}_{\triangle}$ if and only if $A\uparrow$ in $\mathsf{N}^{\uparrow\downarrow}_{\triangle}$.*

2.2. Challenge

The relationship between $\mathsf{N}^{\mathsf{true}}_{\triangle}$ and $\mathsf{N}^{\uparrow\downarrow}_{\triangle}$ in Conjecture 2.2 is considerably more complex than between $\mathsf{N}^{\mathsf{true}}$ and $\mathsf{N}^{\uparrow\downarrow}$ in Proposition 2.1, since $\mathsf{N}^{\mathsf{true}}_{\triangle}$ subsumes $\mathsf{N}^{\uparrow\downarrow}_{\triangle}$ whereas $\mathsf{N}^{\mathsf{true}}$ is separate from $\mathsf{N}^{\uparrow\downarrow}$. In particular, Conjecture 2.2 makes sense only if every proof $\mathcal{D}$ of $A\uparrow$ in $\mathsf{N}^{\uparrow\downarrow}_{\triangle}$ is a refinement of a certain proof $\mathcal{E}$ of A *true* so that $\mathcal{D}$ transforms into $\mathcal{E}$ by replacing some normality or neutrality judgments in it with truth judgments. This requirement holds trivially in $\mathsf{N}^{\uparrow\downarrow}$, but not in $\mathsf{N}^{\uparrow\downarrow}_{\triangle}$ as it stands now. For example, the following proof of $A \supset \triangle A\uparrow$ has no corresponding proof of $A \supset \triangle A$ *true*:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\downarrow}\,{\scriptstyle u}}{A\uparrow}\,{\downarrow\uparrow}}{\triangle A\uparrow}\,\triangle\mathsf{I}_{\uparrow}}{A \supset \triangle A\uparrow}\,\supset\!\mathsf{I}^{u}_{\uparrow} \qquad \longrightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,{\scriptstyle x}}{A\uparrow}\,???}{\triangle A\ \mathit{true}}\,\triangle\mathsf{I}}{A \supset \triangle A\ \mathit{true}}\,\supset\!\mathsf{I}^{x}$$

The problem with $\mathsf{N}^{\mathsf{true}}_{\triangle}$ consists in the rule $\triangle\mathsf{I}_{\uparrow}$: not every normality judgment $A\uparrow$ is eligible as the premise of the rule $\triangle\mathsf{I}_{\uparrow}$, which, however, places no restriction on its premise. To see why, think of the rule $\triangle\mathsf{I}$ as opening up an "inner region" starting from its premise $A\uparrow$ within an "outer region" ending with its conclusion $\triangle A$ *true*:[4]



The two regions are separate from each other because the inner region proves a normality judgment whereas the outer region consists only of truth judgments. Since the rule $\triangle\mathsf{I}_{\uparrow}$ refines the rule $\triangle\mathsf{I}$, its premise cannot reside in the same region as its conclusion. The rule $\triangle\mathsf{I}_{\uparrow}$ in its current form, however, fails to specify that its premise and conclusion reside in separate regions. In the

[4]Here we do not formally define the notion of "region" because we introduce it only to help describe the structure of proofs involving normality judgments.

example above, the hypothesis $\overline{A\downarrow}^{\,u}$ resides in the same region as the conclusion of the rule $\triangle\mathsf{I}_\uparrow$, which implies that the premise of the rule $\triangle\mathsf{I}_\uparrow$ also resides in the same region.

Thus we can imagine that there is an infinite stack of regions and that every judgment in a valid proof resides in a certain unique region, *i.e.*, no conflict arises in assigning a region to each judgment. The region where a judgment resides is determined as follows:

- A hypothesis introduced by the rule $\supset\mathsf{I}$ or $\supset\mathsf{I}_\uparrow$ resides in the same region as the conclusion.
- The premise of the rule $\triangle\mathsf{I}$ or $\triangle\mathsf{I}_\uparrow$ resides in the next inner region.
- A hypothesis introduced by the rule $\triangle\mathsf{E}$ or $\triangle\mathsf{E}_\downarrow$ resides in the next inner region.

Then, for example, an attempt to prove $A \supset \triangle A\uparrow$ ends up with a conflict in assigning a region to the judgment $A\uparrow$:

$$
\begin{array}{ll}
\textit{outer region}\ \{ & \overline{A\downarrow}^{\,u} \\
& \dfrac{}{A\uparrow}\ \downarrow\uparrow \qquad \}\ \textit{outer region? inner region?} \\
& \left.\dfrac{\dfrac{}{\triangle A\uparrow}\ \triangle\mathsf{I}_\uparrow}{A \supset \triangle A\uparrow}\ \supset\mathsf{I}^u_\uparrow\ \right\}\ \textit{outer region}
\end{array}
$$

Another subtle issue with $\mathsf{N}^{\mathsf{true}}_\triangle$ is that a proof of $A\uparrow$ does not necessarily correspond to a normal proof of A *true* in the standard sense, *i.e.*, bottom-up applications of introduction rules connected with top-down applications of elimination rules. The reason is that with the inclusion of the rules $\triangle\mathsf{E}$ and $\triangle\mathsf{E}_\downarrow$, a proof of $A\uparrow$ may use hypotheses of normality judgments, in which case it asserts only the possibility of building a normal proof of A *true*. Such a normal proof can be obtained by substituting an actual normal proof of B *true* for each hypothesis $\overline{B\uparrow}$ in the proof of $A\uparrow$.

Thus the challenge now is to reformulate $\mathsf{N}^{\mathsf{true}}_\triangle$ so that it properly accounts for the relationship between judgments residing in different regions and also clearly distinguishes between normal proofs in the standard sense and normal proofs subject to substitutions. In order to permit judgments residing in different regions, $\mathsf{N}^{\mathsf{true}}_\triangle$ uses two separate contexts in its hypothetical judgments; in order to permit normal proofs subject to substitutions, $\mathsf{N}^{\mathsf{true}}_\triangle$ introduces a weaker form of normality judgments called *semi-normality judgments*.

For technical reasons, we set out to develop a sequent calculus $\mathsf{S}_\triangle$ (Section 3), which, in comparison with a natural deduction system, lends itself better to checking the soundness (or consistency) of the system. It is customary to start with a natural deduction system and then derive a sequent calculus. When designing a new system whose soundness is unclear, however, it is better to consider a sequent calculus before developing a natural deduction system because we need a sequent calculus anyway in order to check its soundness. A cut elimination theorem proves that the system is indeed sound (Section 4). From the sequent calculus, it is routine to derive corresponding natural deduction systems ($\mathsf{N}^{\uparrow\downarrow}_\triangle$ in Section 5 and $\mathsf{N}^{\mathsf{true}}_\triangle$ in 6).

3. Sequent calculus $\mathsf{S}_\triangle$

This section presents a sequent calculus $\mathsf{S}_\triangle$ which augments the sequent calculus for $\mathsf{N}^{\mathsf{true}}$ with the modality $\triangle$. The main obstacle to developing $\mathsf{S}_\triangle$ is to identify a form of sequent that is finite in size, yet expressive enough to allow for a stack of regions of arbitrary depth. Hence we interpret every sequent relative to a certain hypothetical region, which we refer to as a *reference region*.

Let us begin with a sequent $\Gamma \longrightarrow C$ for the sequent calculus for $\mathsf{N}^{\mathsf{true}}$. A in Γ is interpreted as $A\!\downarrow$ and C as $C\!\uparrow$, both in the reference region. Consider a sequent $\Gamma, \triangle A \longrightarrow C$. Analyzing $\triangle A\!\downarrow$ creates $A\!\uparrow$ in the next inner region (as in the rule $\triangle\mathsf{E}_{\downarrow}$), which does not fit into the present form of sequent. Hence we expand the left side of the sequent with a new context Ψ such that A in Ψ is interpreted as a hypothesis of $A\!\uparrow$ belonging to the next inner region. For example, the following rule now makes sense:

$$\frac{\Psi, A; \Gamma, \triangle A \longrightarrow C}{\Psi; \Gamma, \triangle A \longrightarrow C}\ \triangle L$$

The new form of sequent justifies the following rule:

$$\frac{A \text{ atomic}}{\Psi; \Gamma, A \longrightarrow A}\ \mathit{Init}$$

The rule *Init* (for proving *Init*ial sequents), which corresponds to the rule $\downarrow\!\uparrow$ in $\mathsf{N}^{\uparrow\downarrow}$, converts a neutrality judgment $A\!\downarrow$ into a normality judgment $A\!\uparrow$, both in the reference region. It requires A to be an atomic formula, although the requirement can be lifted (see Proposition 3.1).

Now we need a rule that uses hypotheses of normality judgments in Ψ. In designing such a rule, we should distinguish between those proofs that do not use hypotheses in Ψ and thus yield a normality judgment in the standard sense, and those proofs that use hypotheses in Ψ and thus yield a weaker judgment. Without such a distinction, cut elimination would fail because a proof of $\Psi; \Gamma \longrightarrow C$ that uses hypotheses in Ψ may not reveal the structure of a complete proof of $C\!\uparrow$, which is essential to guarantee cut elimination. Thus we are led to use two forms of sequents defined as follows:

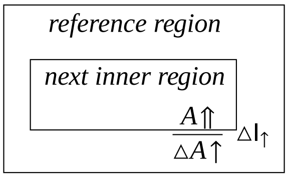
- $\Psi; \Gamma \longrightarrow C$ proves a normality judgment $C\!\uparrow$ in the standard sense. A cut elimination theorem (Theorem 4.1) is to be established for sequents of this form.

- $\Psi; \Gamma \Longrightarrow C$ proves a weaker judgment, namely a semi-normality judgment $C\!\Uparrow$, whose proof may use hypotheses in Ψ. A substitution theorem (Theorem 4.2) is to be established for sequents of this form.

In both forms of sequents, A in Ψ is interpreted as a hypothesis of a semi-normality judgment $A\!\Uparrow$ belonging to the next inner region. In order to use hypotheses in Ψ, we introduce the following rule *Sub* (for *Sub*stituting semi-normality judgments) which moves a semi-normality judgment $A\!\Uparrow$ from the next inner region to the reference region:

$$\frac{}{\Psi, A; \Gamma \Longrightarrow A}\ \mathit{Sub}$$

We use the modality $\triangle$ to internalize *not a normality judgment but a semi-normality judgment* in the next inner region. Using $\triangle$ to internalize a normality judgment is okay, but it renders the system useless because no elimination rule for $\triangle$ is allowed. In fact, adding an elimination rule for $\triangle$ eventually forces us to use $\triangle$ to internalize a semi-normality judgment. Thus, for example, the natural deduction system derived from $\mathsf{S}_{\triangle}$ has the following introduction rule:



An important decision in the design of $\mathsf{S}_{\triangle}$ is to interpret A in a context Ψ as a hypothesis of $A\Uparrow$ belonging to not only the next inner region but also all further inner regions. That is, once a hypothesis of $A\Uparrow$ is added to the next inner region, it is copied to all further inner regions as well, effectively coalescing all inner regions. The rationale is that we wish to design $\mathsf{S}_{\triangle}$ as a reflective system that is aware of its own rule $\dfrac{A\Uparrow}{\triangle A\uparrow}\ \triangle\mathsf{I}_{\uparrow}$, or equivalently, that is aware that $\triangle A\uparrow$ is provable whenever a proof of $A\Uparrow$ exists. The new interpretation of Ψ manifests itself as a proof of $\triangle A \supset \triangle\triangle A\uparrow$ to be shown later. Because of the decision to coalesce all inner regions, $\mathsf{S}_{\triangle}$ needs only a single modality $\triangle$ rather than an infinite tower of modalities $\triangle_1$, $\triangle_2$, $\triangle_3$, $\cdots$.

Thus we use the following definition of $\Psi;\Gamma \longrightarrow C$ and $\Psi;\Gamma \Longrightarrow C$ for $\mathsf{S}_{\triangle}$:

- A in Ψ denotes a hypothesis of $A\Uparrow$ belonging to all inner regions.
- A in Γ denotes a neutrality judgment $A\downarrow$ belonging to the reference region.
- $\Psi;\Gamma \longrightarrow C$ proves $C\uparrow$ in the reference region.
- $\Psi;\Gamma \Longrightarrow C$ proves $C\Uparrow$ in the reference region.

Figure 1 shows the rules in $\mathsf{S}_{\triangle}$. We implicitly identify contexts up to structural equivalence (exchange and contraction). Weakening is also built into the rules *Init*, *Init*$'$, and *Sub*. The requirement on the rule *Init* that A be an atomic formula ensures that any proof of $\Psi;\Gamma \longrightarrow C$ eventually decomposes C into its atomic subformulae. A similar requirement is placed on the rule *Init*$'$, but the rule *Sub* allows us to prove $\Psi;\Gamma \Longrightarrow C$ without analyzing C. The rules $\triangle L$ and $\triangle L'$ analyze $\triangle A\downarrow$ in the reference region and place a hypothesis of $A\Uparrow$ in all inner regions. The rules $\triangle R$ and $\triangle R'$ express that a proof of $\triangle A\uparrow$ or $\triangle A\Uparrow$ requires a proof of $A\Uparrow$ in the next inner region where Ψ continues to be valid (because it contains hypotheses belonging to all inner regions) but Γ from the reference region is no longer valid. The first premise of the rule $\supset L'$ proves $A\uparrow$ instead of $A\Uparrow$, which implies that an analysis of $A \supset B\downarrow$ must be accompanied by a proof of $A\uparrow$. $\mathsf{S}_{\triangle}$ satisfies the subformula property and can be shown to be decidable.

Propositions 3.1 and 3.2 show that A in the rules *Init* and *Init*$'$ can be any formula. The results serve as evidence of the completeness of $\mathsf{S}_{\triangle}$ in the sense that the left rules (such as $\triangle L$ and $\supset L$) are strong enough to guarantee the provability of $C\uparrow$ and $C\Uparrow$ using the right rules (such as $\triangle R$ and $\supset R$) after decomposing C into its atomic subformulae. Proposition 3.3 shows that $\Psi;\Gamma \longrightarrow C$ is stronger than $\Psi;\Gamma \Longrightarrow C$.

Proposition 3.1. *$\cdot;A \longrightarrow A$ is derivable for any formula A.*

Proof. By induction on the structure of A. We show two interesting cases.

$$\dfrac{\dfrac{\dfrac{}{A;\cdot \Longrightarrow A}\ Sub}{A;\triangle A \longrightarrow \triangle A}\ \triangle R}{\cdot;\triangle A \longrightarrow \triangle A}\ \triangle L \qquad \dfrac{\dfrac{\begin{array}{c}\text{IH on } A\\ \cdot;A\supset B, A \longrightarrow A\end{array} \quad \begin{array}{c}\text{IH on } B\\ \cdot;A\supset B, A, B \longrightarrow B\end{array}}{\cdot;A\supset B, A \longrightarrow B}\ \supset L}{\cdot;A\supset B \longrightarrow A\supset B}\ \supset R$$

□

$$\frac{A \text{ atomic}}{\Psi;\Gamma,A \longrightarrow A}\,\mathit{Init} \qquad \frac{A \text{ atomic}}{\Psi;\Gamma,A \Longrightarrow A}\,\mathit{Init}' \qquad \frac{}{\Psi,A;\Gamma \Longrightarrow A}\,\mathit{Sub}$$

$$\frac{\Psi,A;\Gamma,\triangle A \longrightarrow C}{\Psi;\Gamma,\triangle A \longrightarrow C}\,\triangle L \qquad \frac{\Psi;\cdot \Longrightarrow A}{\Psi;\Gamma \longrightarrow \triangle A}\,\triangle R$$

$$\frac{\Psi,A;\Gamma,\triangle A \Longrightarrow C}{\Psi;\Gamma,\triangle A \Longrightarrow C}\,\triangle L' \qquad \frac{\Psi;\cdot \Longrightarrow A}{\Psi;\Gamma \Longrightarrow \triangle A}\,\triangle R'$$

$$\frac{\Psi;\Gamma,A\supset B \longrightarrow A \quad \Psi;\Gamma,A\supset B,B \longrightarrow C}{\Psi;\Gamma,A\supset B \longrightarrow C}\,{\supset}L \qquad \frac{\Psi;\Gamma,A \longrightarrow B}{\Psi;\Gamma \longrightarrow A\supset B}\,{\supset}R$$

$$\frac{\Psi;\Gamma,A\supset B \longrightarrow A \quad \Psi;\Gamma,A\supset B,B \Longrightarrow C}{\Psi;\Gamma,A\supset B \Longrightarrow C}\,{\supset}L' \qquad \frac{\Psi;\Gamma,A \Longrightarrow B}{\Psi;\Gamma \Longrightarrow A\supset B}\,{\supset}R'$$

Figure 1: Sequent calculus $\mathsf{S}_\triangle$

Proposition 3.2. *$\cdot;A \Longrightarrow A$ is derivable for any formula A.*

Proof. By induction on the structure of A. We show two interesting cases.

$$\frac{\dfrac{\dfrac{}{A;\cdot \Longrightarrow A}\,\mathit{Sub}}{A;\triangle A \Longrightarrow \triangle A}\,\triangle R'}{\cdot;\triangle A \Longrightarrow \triangle A}\,\triangle L' \qquad \frac{\dfrac{\overset{\text{Proposition 3.1}}{\cdot;A\supset B,A \longrightarrow A} \quad \overset{\text{IH on } B}{\cdot;A\supset B,A,B \Longrightarrow B}}{\cdot;A\supset B,A \Longrightarrow B}\,{\supset}L'}{\cdot;A\supset B \Longrightarrow A\supset B}\,{\supset}R'$$

□

Proposition 3.3. *If $\Psi;\Gamma \longrightarrow C$, then $\Psi;\Gamma \Longrightarrow C$.*

Proof. By induction on the structure of the proof of $\Psi;\Gamma \longrightarrow C$. □

The converse of Proposition 3.3 does not hold. For example, $A;\cdot \Longrightarrow A$ is provable by the rule *Sub*, but there is no proof of $A;\cdot \longrightarrow A$. Even $\cdot;\Gamma \Longrightarrow C$ does not imply $\cdot;\Gamma \longrightarrow C$, as shown in the following example:

$$\frac{\dfrac{}{A;\triangle A \Longrightarrow A}\,\mathit{Sub}}{\cdot;\triangle A \Longrightarrow A}\,\triangle L' \qquad \frac{\overset{???}{A;\triangle A \longrightarrow A}}{\cdot;\triangle A \longrightarrow A}\,\triangle L$$

The modality $\triangle$ interacts with the implication connective $\supset$ in the following ways. First $\triangle(\triangle A \supset A)\uparrow$ and $\triangle A \supset A\Uparrow$ are provable (because $\cdot;\triangle A \Longrightarrow A$ is provable), but $\triangle A \supset A\uparrow$ is not provable (because $\cdot;\triangle A \longrightarrow A$ is not provable):

$$\frac{\dfrac{\dfrac{\dfrac{}{A;\triangle A \Longrightarrow A}\,\mathit{Sub}}{\cdot;\triangle A \Longrightarrow A}\,\triangle L'}{\cdot;\cdot \Longrightarrow \triangle A \supset A}\,{\supset}R'}{\cdot;\cdot \longrightarrow \triangle(\triangle A \supset A)}\,\triangle R \qquad \frac{\dfrac{\overset{???}{A;\triangle A \longrightarrow A}}{\cdot;\triangle A \longrightarrow A}\,\triangle L}{\cdot;\cdot \longrightarrow \triangle A \supset A}\,{\supset}R$$

Provability of $\triangle A \supset A\Uparrow$ and unprovability of $\triangle A \supset A\uparrow$ conform to the design of the modality $\triangle$: a formula $\triangle A$ internalizes not a normality judgment $A\uparrow$ but a semi-normality judgment $A\Uparrow$, which may use hypotheses of semi-normality judgments and is thus weaker than $A\uparrow$. (Unprovability of

$\triangle A \supset A\uparrow$ implies unprovability of $\triangle A \supset A$ *true* in the natural deduction system $\mathsf{N}_{\triangle}^{\mathsf{true}}$.) Second, although $A \supset \triangle A\uparrow$ is not provable in general, $\triangle A \supset \triangle\triangle A\uparrow$ is provable, which implies that $\mathsf{S}_{\triangle}$ is aware of its own rule $\dfrac{A\Uparrow}{\triangle A\uparrow}\ \triangle\mathsf{I}_{\uparrow}$:

$$\dfrac{\dfrac{\begin{matrix}???\\ \cdot;\cdot \Longrightarrow A\end{matrix}}{\cdot; A \longrightarrow \triangle A}\,\triangle R}{\cdot;\cdot \longrightarrow A \supset \triangle A}\,{\supset}R \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A;\cdot \Longrightarrow A}\,Sub}{A;\cdot \Longrightarrow \triangle A}\,\triangle R'}{A;\triangle A \longrightarrow \triangle\triangle A}\,\triangle R}{\cdot;\triangle A \longrightarrow \triangle\triangle A}\,\triangle L}{\cdot;\cdot \longrightarrow \triangle A \supset \triangle\triangle A}\,{\supset}R$$

Finally $\triangle(A \supset B) \supset (\triangle A \supset \triangle B)\uparrow$ is not provable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}???\\ A \supset B, A;\cdot \Longrightarrow B\end{matrix}}{A \supset B, A;\triangle(A \supset B),\triangle A \longrightarrow \triangle B}\,\triangle R}{A \supset B;\triangle(A \supset B),\triangle A \longrightarrow \triangle B}\,\triangle L}{\cdot;\triangle(A \supset B),\triangle A \longrightarrow \triangle B}\,\triangle L}{\cdot;\triangle(A \supset B) \longrightarrow \triangle A \supset \triangle B}\,{\supset}R}{\cdot;\cdot \longrightarrow \triangle(A \supset B) \supset (\triangle A \supset \triangle B)}\,{\supset}R$$

Intuitively we cannot build a proof of $B\Uparrow$ from hypotheses of $A \supset B\Uparrow$ and $A\Uparrow$. Instead of $A \supset B\Uparrow$, we need $A \supset B\downarrow$, which the assumption of $\triangle(A \supset B)\downarrow$ fails to provide.

Provability of $\triangle(\triangle A \supset A)\uparrow$ implies that the modality $\triangle$ is fundamentally different from the modality $\Box$ in provability logic, which rejects $\Box(\Box A \supset A)$ *true* and admits $(\Box(\Box A \supset A)) \supset \Box A$ *true* as an axiom. Unprovability of $\triangle(A \supset B) \supset (\triangle A \supset \triangle B)\uparrow$ implies that the modality $\triangle$ is fundamentally different from the necessity modality $\Box$ in modal logic, since $\Box(A \supset B) \supset (\Box A \supset \Box B)$ *true* is provable in modal logic like S4.

4. Cut elimination in $\mathsf{S}_{\triangle}$

This section proves cut elimination in $\mathsf{S}_{\triangle}$ which serves as evidence of its soundness. As $\mathsf{S}_{\triangle}$ uses two disjoint contexts in a sequent, we consider two different forms of cut elimination. As usual, the main cut elimination theorem analyzes a proof of $A\uparrow$ to remove a neutrality judgment $A\downarrow$:

Theorem 4.1 (cut elimination)**.**
If $\Psi;\Gamma \longrightarrow A$ *and* $\Psi;\Gamma, A \longrightarrow C$*, then* $\Psi;\Gamma \longrightarrow C$.
If $\Psi;\Gamma \longrightarrow A$ *and* $\Psi;\Gamma, A \Longrightarrow C$*, then* $\Psi;\Gamma \Longrightarrow C$.

In order to remove A in $\Psi, A;\Gamma \longrightarrow C$ or $\Psi, A;\Gamma \Longrightarrow C$, we have to provide a proof of $A\Uparrow$ that can be substituted for a hypothesis of $A\Uparrow$ residing in the next inner region. Hence the following property is called a substitution theorem rather than another cut elimination theorem:

Theorem 4.2 (substitution of semi-normality judgments)**.**
If $\Psi;\cdot \Longrightarrow A$ *and* $\Psi, A;\Gamma \longrightarrow C$*, then* $\Psi;\Gamma \longrightarrow C$.
If $\Psi;\cdot \Longrightarrow A$ *and* $\Psi, A;\Gamma \Longrightarrow C$*, then* $\Psi;\Gamma \Longrightarrow C$.

$$\frac{}{\Psi, A; \Gamma \vdash A\Uparrow}\ \mathsf{Hyp} \qquad \frac{}{\Psi; \Gamma, A \vdash A\downarrow}\ \mathsf{Hyp}'$$

$$\frac{\Psi; \Gamma \vdash A\downarrow}{\Psi; \Gamma \vdash A\uparrow}\ \downarrow\!\uparrow\ (A \text{ atomic}) \qquad \frac{\Psi; \Gamma \vdash A\downarrow}{\Psi; \Gamma \vdash A\Uparrow}\ \downarrow\!\Uparrow\ (A \text{ atomic})$$

$$\frac{\Psi; \cdot \vdash A\Uparrow}{\Psi; \Gamma \vdash \triangle A\uparrow}\ \triangle\mathsf{I}_{\uparrow} \qquad \frac{\Psi; \cdot \vdash A\Uparrow}{\Psi; \Gamma \vdash \triangle A\Uparrow}\ \triangle\mathsf{I}_{\Uparrow}$$

$$\frac{\Psi; \Gamma \vdash \triangle A\downarrow \quad \Psi, A; \Gamma \vdash J}{\Psi; \Gamma \vdash J}\ \triangle\mathsf{E}_{\downarrow}\ (J = C\uparrow \text{ or } C\Uparrow)$$

$$\frac{\Psi; \Gamma, A \vdash B\uparrow}{\Psi; \Gamma \vdash A \supset B\uparrow}\ \supset\!\mathsf{I}_{\uparrow} \quad \frac{\Psi; \Gamma, A \vdash B\Uparrow}{\Psi; \Gamma \vdash A \supset B\Uparrow}\ \supset\!\mathsf{I}_{\Uparrow} \quad \frac{\Psi; \Gamma \vdash A \supset B\downarrow \quad \Psi; \Gamma \vdash A\uparrow}{\Psi; \Gamma \vdash B\downarrow}\ \supset\!\mathsf{E}_{\downarrow}$$

Figure 2: Natural deduction system $\mathsf{N}_{\triangle}^{\uparrow\downarrow}$

Note that $\Psi; \cdot \Longrightarrow A$ can be thought of as proving $A\Uparrow$ in the next inner region because it uses no neutrality judgment and every hypothesis contained in Ψ is assumed to be valid in all inner regions.

We first prove Theorem 4.2 which is used in the proof of Theorem 4.1.

Proof of Theorem 4.2. By simultaneous induction on the structure of the proof of $\Psi, A; \Gamma \longrightarrow C$ and $\Psi, A; \Gamma \Longrightarrow C$. See Appendix for details. □

Proof of Theorem 4.1. By nested induction on the structure of the cut-formula A, the proof $\mathcal{D}$ of $\Psi; \Gamma \longrightarrow A$, and the proof $\mathcal{E}$ of $\Psi; \Gamma, A \longrightarrow C$ or $\Psi; \Gamma, A \Longrightarrow C$. See Appendix for details. □

The proof of Theorem 4.1 illustrates that the rule $\supset L'$ must have $\Psi; \Gamma, A \supset B \longrightarrow A$, instead of $\Psi; \Gamma, A \supset B \Longrightarrow A$, as its first premise. (Otherwise the proof of Theorem 4.1 fails.)

5. Natural deduction system $\mathsf{N}_{\triangle}^{\uparrow\downarrow}$

This section derives a natural deduction system $\mathsf{N}_{\triangle}^{\uparrow\downarrow}$ from the sequent calculus $\mathsf{S}_{\triangle}$. We use a hypothetical judgment of the form $\Psi; \Gamma \vdash J$ with the following assumption:

- A in Ψ is interpreted as a hypothesis of $A\Uparrow$ belonging to the next inner region. As in $\mathsf{S}_{\triangle}$, we assume that it is copied to all further inner regions as well.
- A in Γ is interpreted as a hypothesis of $A\downarrow$ belonging to the reference region.
- J is a semi-normality judgment $C\Uparrow$, a neutrality judgment $C\downarrow$, or a normality judgment $C\uparrow$, all belonging to the reference region.

Figure 2 shows the rules in $\mathsf{N}_{\triangle}^{\uparrow\downarrow}$. Except for the rules Hyp and Hyp' which reflect the definition of the hypothetical judgment $\Psi; \Gamma \vdash J$, each rule has its counterpart in $\mathsf{S}_{\triangle}$. For example, the rules $\downarrow\!\uparrow$ and $\downarrow\!\Uparrow$ correspond to the rules *Init* and *Init*$'$, respectively. An introduction rule (*e.g.*, $\triangle\mathsf{I}_{\uparrow}$) corresponds to a right rule (*e.g.*, $\triangle R$) and an elimination rule (*e.g.*, $\triangle\mathsf{E}_{\downarrow}$) to left rules (*e.g.*, $\triangle L$ and $\triangle L'$).

Proposition 5.1 states a general property of hypothetical judgments in $\mathsf{N}_{\triangle}^{\uparrow\downarrow}$. In the first clause, we prove $\Psi; \cdot \vdash A\Uparrow$, instead of $\Psi; \Gamma \vdash A\Uparrow$, because A in $\Psi, A; \Gamma \vdash J$ denotes $A\Uparrow$ belonging to the next inner region. Theorem 5.4 follows from Lemmas 5.2 and 5.3, and proves that $\mathsf{S}_{\triangle}$ and $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$ are equivalent.

Proposition 5.1.
If $\Psi; \cdot \vdash A\Uparrow$ *and* $\Psi, A; \Gamma \vdash J$*, then* $\Psi; \Gamma \vdash J$.
If $\Psi; \Gamma \vdash A\downarrow$ *and* $\Psi; \Gamma, A \vdash J$*, then* $\Psi; \Gamma \vdash J$.

Proof. By simultaneous induction on the structure of the proof of $\Psi, A; \Gamma \vdash J$ and $\Psi; \Gamma, A \vdash J$. □

Lemma 5.2.
If $\Psi; \Gamma \longrightarrow C$*, then* $\Psi; \Gamma \vdash C\uparrow$.
If $\Psi; \Gamma \Longrightarrow C$*, then* $\Psi; \Gamma \vdash C\Uparrow$.

Proof. By simultaneous induction on the structure of the proof of $\Psi; \Gamma \longrightarrow C$ and $\Psi; \Gamma \Longrightarrow C$. □

Lemma 5.3.
If $\Psi; \Gamma \vdash A\downarrow$*, then* $\Psi; \Gamma, A \longrightarrow C$ *implies* $\Psi; \Gamma \longrightarrow C$.
If $\Psi; \Gamma \vdash A\downarrow$*, then* $\Psi; \Gamma, A \Longrightarrow C$ *implies* $\Psi; \Gamma \Longrightarrow C$.
If $\Psi; \Gamma \vdash A\uparrow$*, then* $\Psi; \Gamma \longrightarrow A$.
If $\Psi; \Gamma \vdash A\Uparrow$*, then* $\Psi; \Gamma \Longrightarrow A$.

Proof. By simultaneous induction on the structure of the proof of $\Psi; \Gamma \vdash A\downarrow$, $\Psi; \Gamma \vdash A\uparrow$, and $\Psi; \Gamma \vdash A\Uparrow$. See Appendix for details. □

Theorem 5.4 ($\mathsf{S}_{\triangle} = \mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$)**.**
$\Psi; \Gamma \longrightarrow C$ *if and only if* $\Psi; \Gamma \vdash C\uparrow$.
$\Psi; \Gamma \Longrightarrow C$ *if and only if* $\Psi; \Gamma \vdash C\Uparrow$.

6. Natural deduction system $\mathsf{N}_{\triangle}^{\mathsf{true}}$

This section presents a natural deduction system $\mathsf{N}_{\triangle}^{\mathsf{true}}$ which is equivalent to $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$, but allows us to deduce truth judgments. Deriving $\mathsf{N}_{\triangle}^{\mathsf{true}}$ from $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$ is analogous to deriving $\mathsf{N}^{\mathsf{true}}$ from $\mathsf{N}^{\upharpoonleft\downharpoonright}$, but more involved because truth judgments coexist with normality judgments and semi-normality judgments in $\mathsf{N}_{\triangle}^{\mathsf{true}}$ whereas only truth judgments exist in $\mathsf{N}^{\mathsf{true}}$.

As it is concerned with deducing truth judgments, $\mathsf{N}_{\triangle}^{\mathsf{true}}$ needs another hypothetical judgment of the form $\Psi; \Gamma \vdash C\ \mathit{true}$. As in $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$, we interpret A in Ψ as a hypothesis of $A\Uparrow$ belonging to the next inner region. A in Γ, however, is interpreted as a hypothesis of $A\ \mathit{true}$ instead of a (stronger) hypothesis of $A\downarrow$. Hence the meaning of Γ in $\Psi; \Gamma \vdash J$ now depends on whether J is a truth judgment or not.

- A in Ψ is interpreted as a hypothesis of $A\Uparrow$ belonging to the next inner region as well as all further inner regions.
- For $J = C\ \mathit{true}$, we interpret A in Γ as a hypothesis of $A\ \mathit{true}$ belonging to the reference region.
- For $J = C\Uparrow$, $C\downarrow$, or $C\uparrow$, we interpret A in Γ as a hypothesis of $A\downarrow$ belonging to the reference region.

$$\frac{}{\Psi; \Gamma, A \vdash A\ \mathit{true}}\ \mathsf{Hyp}''$$

$$\frac{\Psi; \cdot \vdash A \Uparrow}{\Psi; \Gamma \vdash \triangle A\ \mathit{true}}\ \triangle\mathsf{I} \qquad \frac{\Psi; \Gamma \vdash \triangle A\ \mathit{true} \quad \Psi, A; \Gamma \vdash C\ \mathit{true}}{\Psi; \Gamma \vdash C\ \mathit{true}}\ \triangle\mathsf{E}$$

$$\frac{\Psi; \Gamma, A \vdash B\ \mathit{true}}{\Psi; \Gamma \vdash A \supset B\ \mathit{true}}\ \supset\!\mathsf{I} \qquad \frac{\Psi; \Gamma \vdash A \supset B\ \mathit{true} \quad \Psi; \Gamma \vdash A\ \mathit{true}}{\Psi; \Gamma \vdash B\ \mathit{true}}\ \supset\!\mathsf{E}$$

Figure 3: Rules new to the natural deduction system $\mathsf{N}_{\triangle}^{\mathsf{true}}$

- J may be any judgment and belongs to the reference region.

We obtain $\mathsf{N}_{\triangle}^{\mathsf{true}}$ from the rules in $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$ by rewriting both $A\!\uparrow$ and $A\!\downarrow$ as $A\ \mathit{true}$, thereby collapsing the distinction between $A\!\uparrow$ and $A\!\downarrow$. Figure 3 shows the rules in $\mathsf{N}_{\triangle}^{\mathsf{true}}$ obtained from $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$ in this way. The rule Hyp'' expresses that Γ in $\Psi; \Gamma \vdash C\ \mathit{true}$ denotes hypotheses of truth judgments. Note that the rule $\Downarrow\!\Uparrow$ in $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$ does not give rise to a new rule $\frac{\Psi; \Gamma \vdash A\ \mathit{true}}{\Psi; \Gamma \vdash A \Uparrow}\ \mathsf{true}\!\Uparrow$, which does not make sense because Γ in the premise, which denotes hypotheses of truth judgments, is incompatible with Γ in the conclusion, which denotes hypotheses of neutrality judgments.

$\mathsf{N}_{\triangle}^{\mathsf{true}}$ also subsumes $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$ as a subsystem, *i.e.*, it includes all the rules in $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$. Note that the presence of the rule $\triangle\mathsf{I}$ leads $\mathsf{N}_{\triangle}^{\mathsf{true}}$ to include those rules in $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$ for deducing semi-normality judgments. $\Downarrow\!\Uparrow$ is such a rule, whose premise deduces a neutrality judgment. Therefore $\mathsf{N}_{\triangle}^{\mathsf{true}}$ includes those rules in $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$ for deducing neutrality judgments as well. An example of such a rule is $\supset\!\mathsf{E}_{\downarrow}$, whose second premise deduces a normality judgment. Therefore $\mathsf{N}_{\triangle}^{\mathsf{true}}$ also includes those rules in $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$ for deducing normality judgments. Thus $\mathsf{N}_{\triangle}^{\mathsf{true}}$ inherits all the rules from $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$.

Since no rule in Figure 3 deduces a normality or semi-normality judgment while $\mathsf{N}_{\triangle}^{\mathsf{true}}$ subsumes $\mathsf{N}_{\triangle}^{\upharpoonleft\downharpoonright}$ as a subsystem, Theorem 5.4 continues to hold for $\mathsf{N}_{\triangle}^{\mathsf{true}}$. Theorem 6.2 proves the equivalence between $\mathsf{S}_{\triangle}$ and $\mathsf{N}_{\triangle}^{\mathsf{true}}$. In conjunction with Theorem 5.4, it proves the equivalence between normality judgments and truth judgments as stated in Corollary 6.3, which resolves Conjecture 2.2.

Lemma 6.1.
If $\Psi; \Gamma \vdash C\!\uparrow$*, then* $\Psi; \Gamma \vdash C\ \mathit{true}$*.*
If $\Psi; \Gamma \vdash C\!\downarrow$*, then* $\Psi; \Gamma \vdash C\ \mathit{true}$*.*

Proof. By simultaneous induction on the structure of the proof of $\Psi; \Gamma \vdash C\!\uparrow$ and $\Psi; \Gamma \vdash C\!\downarrow$. ☐

Theorem 6.2 ($\mathsf{S}_{\triangle} = \mathsf{N}_{\triangle}^{\mathsf{true}}$)**.** $\Psi; \Gamma \longrightarrow C$ *if and only if* $\Psi; \Gamma \vdash C\ \mathit{true}$*.*

Proof. The *only if* part follows from Theorem 5.4 and Lemma 6.1. The proof of the *if* part proceeds by induction on the structure of the proof of $\Psi; \Gamma \vdash C\ \mathit{true}$. We show two important cases.

Case $\frac{\Psi; \cdot \vdash C' \Uparrow}{\Psi; \Gamma \vdash \triangle C'\ \mathit{true}}\ \triangle\mathsf{I}$ where $C = \triangle C'$

$\Psi; \cdot \Longrightarrow C'$ — from Theorem 5.4 and $\Psi; \cdot \vdash C' \Uparrow$

$\Psi; \Gamma \longrightarrow \triangle C'$ — by the rule $\triangle R$

Case $\frac{\Psi; \Gamma \vdash \triangle A\ \mathit{true} \quad \Psi, A; \Gamma \vdash C\ \mathit{true}}{\Psi; \Gamma \vdash C\ \mathit{true}}\ \triangle\mathsf{E}$

$\Psi, A; \Gamma \longrightarrow C$	by IH on $\Psi, A; \Gamma \vdash C$ *true*
$\Psi, A; \Gamma, \triangle A \longrightarrow C$	by weakening
$\Psi; \Gamma, \triangle A \longrightarrow C$	by the rule $\triangle L$
$\Psi; \Gamma \longrightarrow \triangle A$	by IH on $\Psi; \Gamma \vdash \triangle A$ *true*
$\Psi; \Gamma \longrightarrow C$	by applying Theorem 4.1 to $\Psi; \Gamma \longrightarrow \triangle A$ and $\Psi; \Gamma, \triangle A \longrightarrow C$

□

Corollary 6.3 ($\mathsf{N}_{\triangle}^{\uparrow\downarrow} = \mathsf{N}_{\triangle}^{\mathsf{true}}$)**.** *$\Psi; \Gamma \vdash C\uparrow$ if and only if $\Psi; \Gamma \vdash C$ true.*

We can also explain the relationship between truth judgments and semi-normality judgments. Intuitively a proof of A *true* implies (but does not directly assert) the existence of a proof of a normality judgment $A\uparrow$ whereas a proof of $A\Uparrow$ does not. Hence, whenever A *true* is provable, $A\Uparrow$ is also provable. Suppose $\cdot; \cdot \vdash A$ *true*. By Theorem 6.2, it implies $\cdot; \cdot \longrightarrow A$, which in turn implies $\cdot; \cdot \Longrightarrow A$ by Proposition 3.3. By Theorem 5.4, we conclude $\cdot; \cdot \vdash A\Uparrow$. The converse does not hold, however. For example, $\cdot; \cdot \vdash \triangle A \supset A\Uparrow$ is provable, but $\cdot; \cdot \vdash \triangle A \supset A$ *true* is not:

$$\dfrac{\dfrac{\dfrac{}{\cdot; \triangle A \vdash \triangle A\downarrow}\,\mathsf{Hyp}' \quad \dfrac{}{A; \triangle A \vdash A\Uparrow}\,\mathsf{Hyp}}{\cdot; \triangle A \vdash A\Uparrow}\,\triangle\mathsf{E}_{\downarrow}}{\cdot; \cdot \vdash \triangle A \supset A\Uparrow}\,\supset\!\mathsf{I}_{\Uparrow}$$

$$\dfrac{\dfrac{\dfrac{}{\cdot; \triangle A \vdash \triangle A\ \mathit{true}}\,\mathsf{Hyp}'' \quad \dfrac{???}{A; \triangle A \vdash A\ \mathit{true}}}{\cdot; \triangle A \vdash A\ \mathit{true}}\,\triangle\mathsf{E}}{\cdot; \cdot \vdash \triangle A \supset A\ \mathit{true}}\,\supset\!\mathsf{I}$$

If $A\Uparrow$ is provable while A *true* is not, we have evidence of the truth of A, but its proof does not have a corresponding normal proof. In such a case, we may internalize $A\Uparrow$ within a truth judgment $\triangle A$ *true*, which is guaranteed to have a normal proof. For example, provability of $\triangle A \supset A\Uparrow$ is concisely expressed by a truth judgment $\triangle(\triangle A \supset A)$ *true*.

We close this section by summarizing properties of $\mathsf{N}_{\triangle}^{\mathsf{true}}$ in an axiomatic style:

- $\triangle(\triangle A \supset A)$ *true* is provable (**U**).
- $\triangle A \supset \triangle\triangle A$ *true* is provable (**4**).
- $\triangle(A \supset B) \supset (\triangle A \supset \triangle B)$ *true* is not provable (**K**).
- $\triangle A \supset A$ *true* is not provable (**T**).
- If A *true* is provable, then $\triangle A$ *true* is provable.
- If $\triangle A \supset A$ *true* is provable, then A *true* is provable.

The last two statements are metalogical properties of $\mathsf{N}_{\triangle}^{\mathsf{true}}$, which we prove in Propositions 6.4 and 6.5.

Proposition 6.4. *If A true is provable, then $\triangle A$ true is provable.*

Proof. By Theorem 6.2, it suffices to prove that $\cdot; \cdot \longrightarrow A$ implies $\cdot; \cdot \longrightarrow \triangle A$. Suppose $\cdot; \cdot \longrightarrow A$. By Proposition 3.3, we have $\cdot; \cdot \Longrightarrow A$. By applying the rule $\triangle R$, we obtain $\cdot; \cdot \longrightarrow \triangle A$. □

Proposition 6.5. *If $\triangle A \supset A$ true is provable, then A true is provable.*

Proof. By Theorem 6.2, it suffices to prove that $\cdot;\cdot \longrightarrow \triangle A \supset A$ implies $\cdot;\cdot \longrightarrow A$. The proof of $\cdot;\cdot \longrightarrow \triangle A \supset A$ has the following form:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \cdot;\triangle A \longrightarrow A\end{array}}{\cdot;\cdot \longrightarrow \triangle A \supset A}\;\supset\!R$$

Suppose that $\triangle A$ in $\cdot;\triangle A \longrightarrow A$ is required in the proof $\mathcal{D}$. Then we eventually apply the rule $\triangle L$ to decompose $\triangle A$ and then apply the rule *Sub* as follows:

$$\dfrac{\begin{array}{c}\dfrac{}{\Psi', A; \Gamma' \Longrightarrow A}\;Sub\\ \vdots\;\}\,\mathcal{E}\\ \dfrac{\Psi, A; \triangle A, \Gamma \longrightarrow C}{\Psi; \triangle A, \Gamma \longrightarrow C}\;\triangle L\\ \vdots\;\}\,\mathcal{E}'\\ \cdot;\triangle A \longrightarrow A\end{array}}{\cdot;\cdot \longrightarrow \triangle A \supset A}\;\supset\!R$$

Here we assume that the rule $\triangle L$ is not applied again in $\mathcal{E}$ to decompose $\triangle A$. Note that $\mathcal{E}'$ must contain only applications of the rules $\triangle L$, $\supset\!L$, and $\supset\!R$, and no application of the rule $\triangle R$, in the presence of which we cannot apply the rule $\triangle L$ or $\triangle L'$ to decompose $\triangle A$. Because of the subformula property of the sequent calculus $\mathsf{S}_\triangle$, all formulae in Ψ, Ψ', Γ, and Γ' as well as the formula C are subformulae of A. (We have $\Psi \subseteq \Psi'$, but Γ is not necessarily a subset of Γ'.)

Now observe that no right rule ($\triangle R$, $\triangle R'$, $\supset\!R$, and $\supset\!R'$) appears in $\mathcal{E}$ because C is a subformula of A. Therefore $\mathcal{E}$ may apply only the rules $\triangle L'$ and $\supset\!L'$, which contradicts the assumption that it terminates with a sequent of the form $\Psi''; \Gamma'' \longrightarrow C$ (where $\Psi'' = \Psi, A$ and $\Gamma'' = \triangle A, \Gamma$). Hence $\triangle A$ in $\cdot;\triangle A \longrightarrow A$ is unnecessary in the proof of $\mathcal{D}$ and there exists a proof of $\cdot;\cdot \longrightarrow A$. □

7. Conclusion

We present a system of modal logic that extends a fragment of propositional logic with the implication connective $\supset$ and uses a novel modality $\triangle$ to express the notion of normal proof within the system itself. A sequent calculus is developed to ensure that the system is sound, and then equivalent natural deduction systems are derived. The main obstacle to developing the system is to identify a form of sequent that reflects the self-referential nature of truth judgments and normality judgments. We find that only semi-normality judgments, which are a weaker form of normality judgments, can be internalized within truth judgments using $\triangle$.

Future work includes extending our system to full propositional logic and first-order logic, which do not require additional forms of sequents or hypothetical judgments. As the present work uses a purely proof-theoretic approach, a model-theoretic account of the modality $\triangle$ is another direction to pursue.

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Proof of Theorem 4.2. By simultaneous induction on the structure of the proof of $\Psi, A; \Gamma \longrightarrow C$ and $\Psi, A; \Gamma \Longrightarrow C$.

Case $\dfrac{C \text{ atomic}}{\Psi, A; \Gamma', C \longrightarrow C}\ \mathit{Init}$ where $\Gamma = \Gamma', C$

$\Psi; \Gamma', C \longrightarrow C$ — by rule the *Init*

Case $\dfrac{\Psi, A, B; \Gamma', \triangle B \longrightarrow C}{\Psi, A; \Gamma', \triangle B \longrightarrow C}\ \triangle L$ where $\Gamma = \Gamma', \triangle B$

$\Psi, B; \Gamma', \triangle B \longrightarrow C$ — by IH on the premise

$\Psi; \Gamma', \triangle B \longrightarrow C$ — by the rule $\triangle L$

Case $\dfrac{\Psi, A; \Gamma', B_1 \supset B_2 \longrightarrow B_1 \quad \Psi, A; \Gamma', B_1 \supset B_2, B_2 \longrightarrow C}{\Psi, A; \Gamma', B_1 \supset B_2 \longrightarrow C}\ {\supset}L$

where $\Gamma = \Gamma', B_1 \supset B_2$

$\Psi; \Gamma', B_1 \supset B_2 \longrightarrow B_1$ — by IH on the left premise

$\Psi; \Gamma', B_1 \supset B_2, B_2 \longrightarrow C$ — by IH on the right premise

$\Psi; \Gamma', B_1 \supset B_2 \longrightarrow C$ — by the rule ${\supset}L$

Case $\dfrac{\Psi, A; \cdot \Longrightarrow C'}{\Psi, A; \Gamma \longrightarrow \triangle C'}\ \triangle R$ where $C = \triangle C'$

$\Psi; \cdot \Longrightarrow C'$ — by IH on the premise

$\Psi; \Gamma \longrightarrow \triangle C'$ — by the rule $\triangle R$

Case $\dfrac{\Psi, A; \Gamma, C_1 \longrightarrow C_2}{\Psi, A; \Gamma \longrightarrow C_1 \supset C_2} \supset R$ where $C = C_1 \supset C_2$

$\Psi; \Gamma, C_1 \longrightarrow C_2$ by IH on the premise
$\Psi; \Gamma \longrightarrow C_1 \supset C_2$ by the rule $\supset R$

Case $\dfrac{C \text{ atomic}}{\Psi, A; \Gamma', C \Longrightarrow C} \mathit{Init}'$ where $\Gamma = \Gamma', C$

$\Psi; \Gamma', C \Longrightarrow C$ by rule the Init'

Case $\dfrac{}{\Psi, A; \Gamma \Longrightarrow A} \mathit{Sub}$ where $C = A$

$\Psi; \Gamma \Longrightarrow A$ by weakening $\Psi; \cdot \Longrightarrow A$

Case $\dfrac{}{\Psi', C, A; \Gamma \Longrightarrow C} \mathit{Sub}$ where $C \neq A$ and $\Psi = \Psi', C$

$\Psi', C; \Gamma \Longrightarrow C$ by the rule Sub

Case $\dfrac{\Psi, A, B; \Gamma', \triangle B \Longrightarrow C}{\Psi, A; \Gamma', \triangle B \Longrightarrow C} \triangle L'$ where $\Gamma = \Gamma', \triangle B$

$\Psi, B; \Gamma', \triangle B \Longrightarrow C$ by IH on the premise
$\Psi; \Gamma', \triangle B \Longrightarrow C$ by the rule $\triangle L'$

Case $\dfrac{\Psi, A; \Gamma', B_1 \supset B_2 \longrightarrow B_1 \quad \Psi, A; \Gamma', B_1 \supset B_2, B_2 \Longrightarrow C}{\Psi, A; \Gamma', B_1 \supset B_2 \Longrightarrow C} \supset L'$

where $\Gamma = \Gamma', B_1 \supset B_2$

$\Psi; \Gamma', B_1 \supset B_2 \longrightarrow B_1$ by IH on the left premise
$\Psi; \Gamma', B_1 \supset B_2, B_2 \Longrightarrow C$ by IH on the right premise
$\Psi; \Gamma', B_1 \supset B_2 \Longrightarrow C$ by the rule $\supset L'$

Case $\dfrac{\Psi, A; \cdot \Longrightarrow C'}{\Psi, A; \Gamma \Longrightarrow \triangle C'} \triangle R'$ where $C = \triangle C'$

$\Psi; \cdot \Longrightarrow C'$ by IH on the premise
$\Psi; \Gamma \Longrightarrow \triangle C'$ by the rule $\triangle R'$

Case $\dfrac{\Psi, A; \Gamma, C_1 \Longrightarrow C_2}{\Psi, A; \Gamma \Longrightarrow C_1 \supset C_2} \supset R'$ where $C = C_1 \supset C_2$

$\Psi; \Gamma, C_1 \Longrightarrow C_2$ by IH on the premise
$\Psi; \Gamma \Longrightarrow C_1 \supset C_2$ by the rule $\supset R'$

□

Proof of Theorem 4.1. By nested induction on the structure of the cut-formula A, the proof $\mathcal{D}$ of $\Psi; \Gamma \longrightarrow A$, and the proof $\mathcal{E}$ of $\Psi; \Gamma, A \longrightarrow C$ or $\Psi; \Gamma, A \Longrightarrow C$.

When applying induction hypothesis, we implicitly weaken sequents if necessary. For example, we apply induction hypothesis on a formula A, a proof of $\Psi; \Gamma \longrightarrow A$, and a proof of $\Psi; \Gamma, A, B \longrightarrow C$ to obtain a proof of $\Psi; \Gamma, B \longrightarrow C$, where we implicitly weaken $\Psi; \Gamma \longrightarrow A$ to $\Psi; \Gamma, B \longrightarrow A$. Note that since weakening a sequent does not change the size of its proof, we may still apply induction hypothesis after weakening sequents.

First we consider cases for deducing $\Psi; \Gamma \longrightarrow C$.

Case $\mathcal{D} = \dfrac{A \text{ atomic}}{\Psi; \Gamma', A \longrightarrow A}\ \mathit{Init}$ where $\Gamma = \Gamma', A$

$\Psi; \Gamma', A \longrightarrow C$ — by contraction on $\mathcal{E} :: \Psi; \Gamma', A, A \longrightarrow C$

Case $\mathcal{E} = \dfrac{A \text{ atomic}}{\Psi; \Gamma, A \longrightarrow A}\ \mathit{Init}$ where $C = A$

$\Psi; \Gamma \longrightarrow A$ — from $\mathcal{D}$ and $C = A$

Case $\mathcal{E} = \dfrac{C \text{ atomic}}{\Psi; \Gamma', C, A \longrightarrow C}\ \mathit{Init}$ where $\Gamma = \Gamma', C$

$\Psi; \Gamma', C \longrightarrow C$ — by the rule *Init*

Suppose that A is the principal formula of the last inference rules in both $\mathcal{D}$ and $\mathcal{E}$.

Case $\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}' \\ \Psi; \cdot \Longrightarrow A'\end{matrix}}{\Psi; \Gamma \longrightarrow \triangle A'}\ \triangle R$ and $\mathcal{E} = \dfrac{\begin{matrix}\mathcal{E}' \\ \Psi, A'; \Gamma, \triangle A' \longrightarrow C\end{matrix}}{\Psi; \Gamma, \triangle A' \longrightarrow C}\ \triangle L$

where $A = \triangle A'$

$\mathcal{E}'' :: \Psi, A'; \Gamma \longrightarrow C$ — by IH on $\triangle A'$, $\mathcal{D}$, and $\mathcal{E}'$

$\Psi; \Gamma \longrightarrow C$ — by applying Theorem 4.2 to $\mathcal{D}'$ and $\mathcal{E}''$

Case $\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}' \\ \Psi; \Gamma, A_1 \longrightarrow A_2\end{matrix}}{\Psi; \Gamma \longrightarrow A_1 \supset A_2}\ {\supset}R$

and $\mathcal{E} = \dfrac{\begin{matrix}\mathcal{E}_1 \\ \Psi; \Gamma, A_1 \supset A_2 \longrightarrow A_1\end{matrix} \quad \begin{matrix}\mathcal{E}_2 \\ \Psi; \Gamma, A_1 \supset A_2, A_2 \longrightarrow C\end{matrix}}{\Psi; \Gamma, A_1 \supset A_2 \longrightarrow C}\ {\supset}L$

where $A = A_1 \supset A_2$

$\mathcal{E}_1' :: \Psi; \Gamma \longrightarrow A_1$ — by IH on $A_1 \supset A_2$, $\mathcal{D}$, and $\mathcal{E}_1$

$\mathcal{D}'' :: \Psi; \Gamma \longrightarrow A_2$ — by IH on A_1, $\mathcal{E}_1'$, and $\mathcal{D}'$

$\mathcal{E}_2' :: \Psi; \Gamma, A_2 \longrightarrow C$ — by IH on $A_1 \supset A_2$, $\mathcal{D}$, and $\mathcal{E}_2$

$\Psi; \Gamma \longrightarrow C$ — by IH on A_2, D'', and $\mathcal{E}_2'$

Suppose that A is not the principal formula of the last inference rule in $\mathcal{D}$.

Case $\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}' \\ \Psi, B; \Gamma', \triangle B \longrightarrow A\end{matrix}}{\Psi; \Gamma', \triangle B \longrightarrow A}\ \triangle L$ where $\Gamma = \Gamma', \triangle B$

$\Psi, B; \Gamma', \triangle B \longrightarrow C$ — by IH on A, $\mathcal{D}'$, and $\mathcal{E}$

$\Psi; \Gamma', \triangle B \longrightarrow C$ — by the rule $\triangle L$

Case $\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Psi; \Gamma', B_1 \supset B_2 \longrightarrow B_1\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \Psi; \Gamma', B_1 \supset B_2, B_2 \longrightarrow A\end{matrix}}{\Psi; \Gamma', B_1 \supset B_2 \longrightarrow A}\ {\supset}L$

where $\Gamma = \Gamma', B_1 \supset B_2$

$\mathcal{E}' :: \Psi; \Gamma', B_1 \supset B_2, B_2 \longrightarrow C$ — by IH on A, $\mathcal{D}_2$, and $\mathcal{E}$

$\Psi; \Gamma', B_1 \supset B_2 \longrightarrow C$ — by the rule ${\supset}L$ on $\mathcal{D}_1$ and $\mathcal{E}'$

Suppose that A is not the principal formula of the last inference rule in $\mathcal{E}$.

Case $\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Psi, B; \Gamma', \triangle B, A \longrightarrow C}}{\Psi; \Gamma', \triangle B, A \longrightarrow C} \triangle L$ where $\Gamma = \Gamma', \triangle B$

$\Psi, B; \Gamma', \triangle B \longrightarrow C$ by IH on A, $\mathcal{D}$, and $\mathcal{E}'$

$\Psi; \Gamma', \triangle B \longrightarrow C$ by the rule $\triangle L$

Case $\mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Psi; \Gamma', B_1 \supset B_2, A \longrightarrow B_1} \quad \overset{\mathcal{E}_2}{\Psi; \Gamma', B_1 \supset B_2, A, B_2 \longrightarrow C}}{\Psi; \Gamma', B_1 \supset B_2, A \longrightarrow C} \supset L$

where $\Gamma = \Gamma', B_1 \supset B_2$

$\mathcal{E}_1' :: \Psi; \Gamma', B_1 \supset B_2 \longrightarrow B_1$ by IH on A, $\mathcal{D}$, and $\mathcal{E}_1$

$\mathcal{E}_2' :: \Psi; \Gamma', B_1 \supset B_2, B_2 \longrightarrow C$ by IH on A, $\mathcal{D}$, and $\mathcal{E}_2$

$\Psi; \Gamma', B_1 \supset B_2 \longrightarrow C$ by the rule $\supset L$ on $\mathcal{E}_1'$ and $\mathcal{E}_2'$

Case $\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Psi; \cdot \Longrightarrow C'}}{\Psi; \Gamma, A \longrightarrow \triangle C'} \triangle R$ where $C = \triangle C'$

$\Psi; \Gamma \longrightarrow \triangle C'$ by the rule $\triangle R$ on $\mathcal{E}'$

Case $\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Psi; \Gamma, A, C_1 \longrightarrow C_2}}{\Psi; \Gamma, A \longrightarrow C_1 \supset C_2} \supset R$ where $C = C_1 \supset C_2$

$\Psi; \Gamma, C_1 \longrightarrow C_2$ by IH on A, $\mathcal{D}$, and $\mathcal{E}'$

$\Psi; \Gamma \longrightarrow C_1 \supset C_2$ by the rule $\supset R$

Next we consider cases for deducing $\Psi; \Gamma \Longrightarrow C$.

Case $\mathcal{D} = \dfrac{A \text{ atomic}}{\Psi; \Gamma', A \longrightarrow A}\ \mathit{Init}$ where $\Gamma = \Gamma', A$

$\Psi; \Gamma', A \Longrightarrow C$ by contraction on $\mathcal{E} :: \Psi; \Gamma', A, A \Longrightarrow C$

Case $\mathcal{E} = \dfrac{A \text{ atomic}}{\Psi; \Gamma, A \Longrightarrow A}\ \mathit{Init}'$ where $C = A$

$\Psi; \Gamma \Longrightarrow A$ by applying Proposition 3.3 to $\mathcal{D}$

Case $\mathcal{E} = \dfrac{C \text{ atomic}}{\Psi; \Gamma', C, A \Longrightarrow C}\ \mathit{Init}'$ where $\Gamma = \Gamma', C$

$\Psi; \Gamma', C \Longrightarrow C$ by the rule *Init*

Case $\mathcal{E} = \dfrac{}{\Psi', C; \Gamma, A \Longrightarrow C}\ \mathit{Sub}$ where $\Psi = \Psi', C$

$\Psi', C; \Gamma \Longrightarrow C$ by the rule *Sub*

Suppose that A is the principal formula of the last inference rules in both $\mathcal{D}$ and $\mathcal{E}$.

Case $\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Psi; \cdot \Longrightarrow A'}}{\Psi; \Gamma \longrightarrow \triangle A'} \triangle R$ and $\mathcal{E} = \dfrac{\overset{\mathcal{E}'}{\Psi, A'; \Gamma, \triangle A' \Longrightarrow C}}{\Psi; \Gamma, \triangle A' \Longrightarrow C} \triangle L'$

where $A = \triangle A'$

$\mathcal{E}'' : \Psi, A'; \Gamma \Longrightarrow C$ by IH on $\triangle A'$, $\mathcal{D}$, and $\mathcal{E}'$

$\Psi; \Gamma \Longrightarrow C$ by applying Theorem 4.2 to $\mathcal{D}'$ and $\mathcal{E}''$

$$\text{Case } \mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Psi;\Gamma, A_1 \longrightarrow A_2}}{\Psi;\Gamma \longrightarrow A_1 \supset A_2} \supset\! R$$

$$\text{and } \mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Psi;\Gamma, A_1 \supset A_2 \longrightarrow A_1} \quad \overset{\mathcal{E}_2}{\Psi;\Gamma, A_1 \supset A_2, A_2 \Longrightarrow C}}{\Psi;\Gamma, A_1 \supset A_2 \Longrightarrow C} \supset\! L'$$

where $A = A_1 \supset A_2$

$\mathcal{E}_1' :: \Psi;\Gamma \longrightarrow A_1$ — by IH on $A_1 \supset A_2$, $\mathcal{D}$, and $\mathcal{E}_1$
$\mathcal{D}' :: \Psi;\Gamma \longrightarrow A_2$ — by IH on A_1, $\mathcal{E}_1'$, and $\mathcal{D}'$
$\mathcal{E}_2' :: \Psi;\Gamma, A_2 \Longrightarrow C$ — by IH on $A_1 \supset A_2$, $\mathcal{D}$, and $\mathcal{E}_2$
$\Psi;\Gamma \Longrightarrow C$ — by IH on A_2, $\mathcal{D}'$, and $\mathcal{E}_2'$

We omit all remaining cases which are similar to those cases for deducing $\Psi;\Gamma \longrightarrow C$.

□

Proof of Lemma 5.3. By simultaneous induction on the structure of the proof $\mathcal{D}$ of $\Psi;\Gamma \vdash A\!\downarrow$, $\Psi;\Gamma \vdash A\!\uparrow$, and $\Psi;\Gamma \vdash A\!\Uparrow$.

$$\text{Case } \mathcal{D} = \dfrac{}{\Psi', A;\Gamma \vdash A\!\Uparrow}\ \mathsf{Hyp} \quad \text{where } \Psi = \Psi', A$$

$\Psi', A;\Gamma \Longrightarrow A$ — by the rule *Sub*

$$\text{Case } \mathcal{D} = \dfrac{}{\Psi;\Gamma', A \vdash A\!\downarrow}\ \mathsf{Hyp}' \quad \text{where } \Gamma = \Gamma', A$$

$\Psi;\Gamma', A, A \longrightarrow C$ — assumption
$\Psi;\Gamma', A \longrightarrow C$ — by contraction on $\Psi;\Gamma', A, A \longrightarrow C$

$\Psi;\Gamma', A, A \Longrightarrow C$ — assumption
$\Psi;\Gamma', A \Longrightarrow C$ — by contraction on $\Psi;\Gamma', A, A \Longrightarrow C$

$$\text{Case } \mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Psi;\Gamma \vdash A\!\downarrow}}{\Psi;\Gamma \vdash A\!\uparrow} \downarrow\!\uparrow (A\ \textit{atomic})$$

$\mathcal{E} :: \Psi;\Gamma, A \longrightarrow A$ — by the rule *Init*
$\Psi;\Gamma \longrightarrow A$ — by IH on $\mathcal{D}'$ with $\mathcal{E}$

$$\text{Case } \mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Psi;\Gamma \vdash A\!\downarrow}}{\Psi;\Gamma \vdash A\!\Uparrow} \downarrow\!\Uparrow (A\ \textit{atomic})$$

$\mathcal{E} :: \Psi;\Gamma, A \Longrightarrow A$ — by the rule *Init′*
$\Psi;\Gamma \Longrightarrow A$ — by IH on $\mathcal{D}'$ with $\mathcal{E}$

$$\text{Case } \mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Psi;\cdot \vdash A'\!\Uparrow}}{\Psi;\Gamma \vdash \triangle A'\!\uparrow} \triangle\mathsf{I}_{\uparrow} \quad \text{where } A = \triangle A'$$

$\mathcal{E} :: \Psi;\cdot \Longrightarrow A'$ — by IH on $\mathcal{D}'$
$\Psi;\Gamma \longrightarrow \triangle A'$ — by the rule $\triangle R$ on $\mathcal{E}$

$$\text{Case } \mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Psi;\cdot \vdash A'\!\Uparrow}}{\Psi;\Gamma \vdash \triangle A'\!\Uparrow} \triangle\mathsf{I}_{\Uparrow} \quad \text{where } A = \triangle A'$$

$\mathcal{E} :: \Psi; \cdot \Longrightarrow A'$ by IH on $\mathcal{D}'$
$\Psi; \Gamma \Longrightarrow \triangle A'$ by the rule $\triangle R'$ on $\mathcal{E}$

Case $\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Psi; \Gamma \vdash \triangle B\downarrow} \quad \overset{\mathcal{D}_2}{\Psi, B; \Gamma \vdash A\uparrow}}{\Psi; \Gamma \vdash A\uparrow} \triangle\mathsf{E}_{\downarrow}$

$\mathcal{E} :: \Psi, B; \Gamma \longrightarrow A$ by IH on $\mathcal{D}_2$
$\mathcal{E}' :: \Psi, B; \Gamma, \triangle B \longrightarrow A$ by weakening $\mathcal{E}$
$\mathcal{E}'' :: \Psi; \Gamma, \triangle B \longrightarrow A$ by the rule $\triangle L$ on $\mathcal{E}'$
$\Psi; \Gamma \longrightarrow A$ by IH on D_1 with $\mathcal{E}''$

Case $\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Psi; \Gamma \vdash \triangle B\downarrow} \quad \overset{\mathcal{D}_2}{\Psi, B; \Gamma \vdash A\Uparrow}}{\Psi; \Gamma \vdash A\Uparrow} \triangle\mathsf{E}_{\downarrow}$

$\mathcal{E} :: \Psi, B; \Gamma \Longrightarrow A$ by IH on $\mathcal{D}_2$
$\mathcal{E}' :: \Psi, B; \Gamma, \triangle B \Longrightarrow A$ by weakening $\mathcal{E}$
$\mathcal{E}'' :: \Psi; \Gamma, \triangle B \Longrightarrow A$ by the rule $\triangle L'$ on $\mathcal{E}'$
$\Psi; \Gamma \Longrightarrow A$ by IH on $\mathcal{D}_1$ with $\mathcal{E}''$

Case $\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Psi; \Gamma, A_1 \vdash A_2\uparrow}}{\Psi; \Gamma \vdash A_1 \supset A_2\uparrow} \supset\mathsf{I}_{\uparrow}$ where $A = A_1 \supset A_2$

$\mathcal{E} :: \Psi; \Gamma, A_1 \longrightarrow A_2$ by IH on $\mathcal{D}'$
$\Psi; \Gamma \longrightarrow A_1 \supset A_2$ by the rule $\supset R$ on $\mathcal{E}$

Case $\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Psi; \Gamma, A_1 \vdash A_2\Uparrow}}{\Psi; \Gamma \vdash A_1 \supset A_2\Uparrow} \supset\mathsf{I}_{\Uparrow}$ where $A = A_1 \supset A_2$

$\mathcal{E} :: \Psi; \Gamma, A_1 \Longrightarrow A_2$ by IH on $\mathcal{D}'$
$\Psi; \Gamma \Longrightarrow A_1 \supset A_2$ by the rule $\supset R'$ on $\mathcal{E}$

Case $\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Psi; \Gamma \vdash B \supset A\downarrow} \quad \overset{\mathcal{D}_2}{\Psi; \Gamma \vdash B\uparrow}}{\Psi; \Gamma \vdash A\downarrow} \supset\mathsf{E}_{\downarrow}$

$\mathcal{E}_1 :: \Psi; \Gamma \longrightarrow B$ by IH on $\mathcal{D}_2$
$\mathcal{E}_1' :: \Psi; \Gamma, B \supset A \longrightarrow B$ by weakening $\mathcal{E}_1$
$\mathcal{E}_2 :: \Psi; \Gamma, A \longrightarrow C$ assumption
$\mathcal{E}_2' :: \Psi; \Gamma, B \supset A, A \longrightarrow C$ by weakening $\mathcal{E}_2$
$\mathcal{E}_3 :: \Psi; \Gamma, B \supset A \longrightarrow C$ by the rule $\supset L$ on $\mathcal{E}_1'$ and $\mathcal{E}_2'$
$\Psi; \Gamma \longrightarrow C$ by IH on $\mathcal{D}_1$ with $\mathcal{E}_3$

$\mathcal{E}_1 :: \Psi; \Gamma \longrightarrow B$ by IH on $\mathcal{D}_2$
$\mathcal{E}_1' :: \Psi; \Gamma, B \supset A \longrightarrow B$ by weakening $\mathcal{E}_1$
$\mathcal{E}_2 :: \Psi; \Gamma, A \Longrightarrow C$ assumption
$\mathcal{E}_2' :: \Psi; \Gamma, B \supset A, A \Longrightarrow C$ by weakening $\mathcal{E}_2$
$\mathcal{E}_3 :: \Psi; \Gamma, B \supset A \Longrightarrow C$ by the rule $\supset L'$ on $\mathcal{E}_1'$ and $\mathcal{E}_2'$
$\Psi; \Gamma \Longrightarrow C$ by IH on $\mathcal{D}_1$ with $\mathcal{E}_3$

□