Name: Hemos ID:

CSE-321 Programming Languages 2007
Quiz 2 — Sample Solution

	Problem 1	Problem 2	Problem 3	Total
Score				
Max	30	30	40	100

Throughout this quiz, we will use the following fragment of the simply typed λ-calculus. (We do not consider base types.)

$$
\begin{array}{rrcl}
\textsf{type} & A & ::= & P \mid A \rightarrow A \\
\textsf{base type} & P & & \\
\textsf{expression} & e & ::= & x \mid \lambda x\!:\!A.\,e \mid e\ e \\
\textsf{value} & v & ::= & \lambda x\!:\!A.\,e \\
\textsf{typing context} & \Gamma & ::= & \cdot \mid \Gamma, x : A
\end{array}
$$

$$
\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ \textsf{Var} \qquad
\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x\!:\!A.\,e : A \rightarrow B}\ {\rightarrow}\textsf{I} \qquad
\frac{\Gamma \vdash e : A \rightarrow B \quad \Gamma \vdash e' : A}{\Gamma \vdash e\ e' : B}\ {\rightarrow}\textsf{E}
$$

$$
\frac{e_1 \mapsto e_1'}{e_1\ e_2 \mapsto e_1'\ e_2}\ \mathit{Lam} \qquad
\frac{e_2 \mapsto e_2'}{(\lambda x\!:\!A.\,e)\ e_2 \mapsto (\lambda x\!:\!A.\,e)\ e_2'}\ \mathit{Arg} \qquad
\frac{}{(\lambda x\!:\!A.\,e)\ v \mapsto [v/x]e}\ \mathit{App}
$$

1 Weakening [30 pts]

Fill in the blank to complete the proof. You may use Proposition 1.1 in the proof of Proposition 1.2.

Proposition 1.1 (Permutation). *If $\Gamma \vdash e : A$ and Γ' is a permutation of Γ, then $\Gamma' \vdash e : A$.*

Proposition 1.2 (Weakening). *If $\Gamma \vdash e : C$, then $\Gamma, x : A \vdash e : C$.*

Proof. By rule induction on the judgment $\Gamma \vdash e : C$. We consider three cases.

Case $\dfrac{y : C \in \Gamma}{\Gamma \vdash y : C}\,\mathsf{Var}$ where $e = y$:

$y : C \in \Gamma, x : A$ from $\underline{y : C \in \Gamma}$

$\underline{\Gamma, x : A \vdash y : C}$ by the rule Var

Case $\dfrac{\Gamma, y : C_1 \vdash e' : C_2}{\Gamma \vdash \lambda y{:}C_1.\, e' : C_1 \rightarrow C_2}\,{\rightarrow}\mathsf{I}$ where $e = \lambda y{:}C_1.\, e'$ and $C = C_1 \rightarrow C_2$:

$\underline{\Gamma, y : C_1, x : A \vdash e' : C_2}$ by induction hypothesis

$\Gamma, x : A, y : C_1 \vdash e' : C_2$ by Proposition 1.1

$\underline{\Gamma, x : A \vdash \lambda y{:}C_1.\, e' : C_1 \rightarrow C_2}$ by the rule ${\rightarrow}\mathsf{I}$

Case $\dfrac{\Gamma \vdash e_1 : B \rightarrow C \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1\ e_2 : C}\,{\rightarrow}\mathsf{E}$ where $e = e_1\ e_2$:

$\underline{\Gamma, x : A \vdash e_1 : B \rightarrow C}$ by induction hypothesis on $\underline{\Gamma \vdash e_1 : B \rightarrow C}$

$\underline{\Gamma, x : A \vdash e_2 : B}$ by induction hypothesis on $\underline{\Gamma \vdash e_2 : B}$

$\Gamma, x : A \vdash e_1\ e_2 : C$ by <u>the rule ${\rightarrow}\mathsf{E}$</u>

$\square$

2 Substitution [30 pts]

Fill in the blank to complete the proof.

Lemma 2.1 (Substitution). *If* $\Gamma \vdash e : A$ *and* $\Gamma, x : A \vdash e' : C$*, then* $\Gamma \vdash [e/x]e' : C$*.*

Proof. By rule induction on the judgment $\Gamma, x : A \vdash e' : C$. We consider three cases. In the third case, we implicitly use Proposition 1.1 (Permutation).

Case $\dfrac{y : C \in \Gamma, x : A}{\Gamma, x : A \vdash y : C}\,\mathsf{Var}$ where $e' = y$ and $y : C \in \Gamma$:

$\Gamma \vdash y : C$ — from $y : C \in \Gamma$

$[e/x]y = y$ — from $x \neq y$

$\underline{\Gamma \vdash [e/x]y : C}$

Case $\dfrac{}{\Gamma, x : A \vdash x : A}\,\mathsf{Var}$ where $e' = x$ and $C = A$:

$\underline{\Gamma \vdash e : A}$ — assumption

$\underline{\Gamma \vdash [e/x]x : A}$ — $[e/x]x = e$

Case $\dfrac{\Gamma, x : A, y : C_1 \vdash e'' : C_2}{\Gamma, x : A \vdash \lambda y{:}C_1.\, e'' : C_1 \to C_2}\,{\to}\mathsf{I}$ where $e' = \lambda y{:}C_1.\, e''$ and $C = C_1 \to C_2$:

$\underline{\Gamma, y : C_1 \vdash [e/x]e'' : C_2}$ — by induction hypothesis

$\underline{\Gamma \vdash \lambda y{:}C_1.\, [e/x]e'' : C_1 \to C_2}$ — by the rule ${\to}\mathsf{I}$

$[e/x]\lambda y{:}C_1.\, e'' = \underline{\lambda y{:}C_1.\, [e/x]e''}$ — from $y \notin FV(e)$ and $x \neq y$

$\underline{\Gamma \vdash [e/x]\lambda y{:}C_1.\, e'' : C_1 \to C_2}$

$\square$

3 Type preservation [40 pts]

Fill in the blank. You may use Lemmas 2.1 (Substitution) and 3.1 (Inversion).

Lemma 3.1 (Inversion). *Suppose* $\Gamma \vdash e : C$.
If $e = x$, *then* $x : C \in \Gamma$.
If $e = \lambda x{:}A.\, e'$, *then* $C = A \to B$ *and* $\Gamma, x : A \vdash e' : B$ *for some type* B.
If $e = e_1\ e_2$, *then* $\Gamma \vdash e_1 : A \to C$ *and* $\Gamma \vdash e_2 : A$ *for some type* A.

Theorem 3.2 (Type preservation). *If* $\Gamma \vdash e : A$ *and* $e \mapsto e'$, *then* $\Gamma \vdash e' : A$.

Proof. By rule induction on the judgment $e \mapsto e'$.

Case $\dfrac{e_1 \mapsto e_1'}{e_1\ e_2 \mapsto e_1'\ e_2}\ \mathit{Lam}$

$\Gamma \vdash e_1\ e_2 : A$ — assumption

$\underline{\Gamma \vdash e_1 : B \to A}$ and $\underline{\Gamma \vdash e_2 : B}$ for some type B — by Lemma 3.1

$\underline{\Gamma \vdash e_1' : B \to A}$ — by induction hypothesis on $e_1 \mapsto e_1'$ with $\underline{\Gamma \vdash e_1 : B \to A}$

$\Gamma \vdash e_1'\ e_2 : A$ — from $\underline{\dfrac{\Gamma \vdash e_1' : B \to A \quad \Gamma \vdash e_2 : B}{\Gamma \vdash e_1'\ e_2 : A} \to\mathsf{E}}$

Case $\dfrac{e_2 \mapsto e_2'}{(\lambda x{:}B.\, e_1')\ e_2 \mapsto (\lambda x{:}B.\, e_1')\ e_2'}\ \mathit{Arg}$

$\Gamma \vdash (\lambda x{:}B.\, e_1')\ e_2 : A$ — assumption

$\underline{\Gamma \vdash \lambda x{:}B.\, e_1' : B \to A}$ and $\underline{\Gamma \vdash e_2 : B}$ — by Lemma 3.1

$\underline{\Gamma \vdash e_2' : B}$ — by induction hypothesis on $e_2 \mapsto e_2'$ with

$\underline{\Gamma \vdash e_2 : B}$

$\Gamma \vdash (\lambda x{:}B.\, e_1')\ e_2' : A$ — from $\underline{\dfrac{\Gamma \vdash \lambda x{:}B.\, e_1' : B \to A \quad \Gamma \vdash e_2' : B}{\Gamma \vdash (\lambda x{:}B.\, e_1')\ e_2' : A} \to\mathsf{E}}$

Case $\dfrac{}{(\lambda x{:}B.\, e_1')\ v \mapsto [v/x]e_1'}\ \mathit{App}$

$\underline{\Gamma \vdash (\lambda x{:}B.\, e_1')\ v : A}$ — assumption

$\underline{\Gamma \vdash \lambda x{:}B.\, e_1' : B \to A}$ and $\underline{\Gamma \vdash v : B}$ — by Lemma 3.1

$\Gamma, x : B \vdash e_1' : A$ — by Lemma 3.1 on $\underline{\Gamma \vdash \lambda x{:}B.\, e_1' : B \to A}$

$\underline{\Gamma \vdash [v/x]e_1' : A}$ — by applying Lemma 2.1 to $\underline{\Gamma \vdash v : B}$ and $\Gamma, x : B \vdash e_1' : A$