Chapter 1

Recursive Types

In programming in a practical functional language, there often arises a need for recursive data structures (or inductive data structures) whose components are data structures of the same kind but of smaller size. For example, a tree is a recursive data structure because children of the root node are smaller trees of the same kind. We may even think of natural numbers as a recursive data structure because a non-zero natural number can be expressed as a successor of another natural number.

The type system developed so far, however, cannot account for recursive data structures. Intuitively types for recursive data structures require recursive definitions at the level of types, but the previous type system does not provide such a language construct. (Recursive definitions at the level of expressions can be expressed using the fixed point construct.) This chapter introduces a new language construct for declaring recursive types which express recursive definitions at the level of types.

With recursive types, we can declare types for recursive data structures. For example, we declare a recursive type ntree for binary trees of natural numbers (of type nat) with the following recursive definition:

$$\text{ntree} \triangleq \text{nat} + (\text{ntree} \times \text{ntree})$$

The definition says that ntree is either a single natural number of type nat (corresponding to leaf nodes) or two such binary trees of type ntree (corresponding to internal nodes).

There are two approaches to formalizing recursive types: equi-recursive and iso-recursive approaches which differ in the interpretation of $\cong$ in recursive definitions of types. Under the equi-recursive approach, $\cong$ stands for an equality relation. For example, the recursive definition of ntree specifies that ntree and nat + (ntree × ntree) are equal and thus interchangeable: ntree is automatically (i.e., without the intervention of programmers) converted to nat + (ntree × ntree) and vice versa whenever necessary to make a given expression typecheck. Under the iso-recursive approach, $\cong$ stands for an isomorphic relation: two types in a recursive definition cannot be identified, but can be converted to each other by certain functions. For example, the recursive definition of ntree implicitly declares two functions for converting between ntree and nat + (ntree × ntree):

$$\text{fold}_{\text{ntree}} : \text{nat} + (\text{ntree} \times \text{ntree}) \rightarrow \text{ntree}$$
$$\text{unfold}_{\text{ntree}} : \text{ntree} \rightarrow \text{nat} + (\text{ntree} \times \text{ntree})$$

To create a value of type ntree, we first create a value of type nat + (ntree × ntree) and then apply function fold_{ntree}; to analyze a value of type ntree, we first apply function unfold_{ntree} and then analyze the resultant value using a case expression.

Below we formalize recursive types under the iso-recursive approach. We will also see that SML uses the iso-recursive approach to deal with datatype declarations.

1.1 Definition

Consider the recursive definition of ntree. We may think of ntree as the solution to the following equation where $\alpha$ is a type variable standing for “any type” as in SML:

$$\alpha \cong \text{nat} + (\alpha \times \alpha)$$
Since substituting ntree for \( \alpha \) yields the original recursive definition of ntree, ntree is indeed the solution to the above equation. We choose to write \( \mu \alpha . \text{nat} + (\alpha \times \alpha) \) for the solution to the above equation where \( \alpha \) is a fresh type variable. Then we can redefine ntree as follows:

\[
\text{n} \text{tree} = \mu \alpha . \text{nat} + (\alpha \times \alpha)
\]

Generalizing the example of ntree, we use a recursive type \( \mu \alpha . A \) for the solution to the equation \( \alpha \cong A \) where \( A \) may contain occurrences of type variable \( \alpha \):

\[
\text{type} \quad A ::= \cdots \mid \alpha \mid \mu \alpha . A
\]

The intuition is that \( C = \mu \alpha . A \) means \( C \cong [C/\alpha]A \). For example, \( \text{n} \text{tree} = \mu \alpha . \text{nat} + (\alpha \times \alpha) \) means \( \text{n} \text{tree} \cong \text{nat} + (\text{n} \text{tree} \times \text{n} \text{tree}) \). Since \( \mu \alpha . A \) declares a fresh type variable \( \alpha \) which is valid only within \( A \), not every recursive type qualifies as a valid type. For example, \( \mu \alpha . \alpha + \beta \) is not a valid recursive type unless it is part of another recursive type declaring type variable \( \beta \). In order to be able to check the validity of a given recursive type, we define a typing context as an ordered set of type bindings and type declarations:

\[
\text{typing context} \quad \Gamma ::= \cdot \mid \alpha \mid \Gamma, x : A \mid \Gamma, \alpha \text{ type}
\]

We use a new judgment \( \Gamma \vdash A \text{ type} \), called a type judgment, to check that \( A \) is a valid type under typing context \( \Gamma \):

\[
\begin{align*}
\alpha \text{ type} \in \Gamma & \quad \text{TyVar} \\
\Gamma, \alpha \text{ type} \vdash A \text{ type} & \quad \text{Ty} \\
\Gamma \vdash \mu \alpha . A \text{ type} & \quad \text{Ty}\mu
\end{align*}
\]

Given a recursive type \( C = \mu \alpha . A \), we need to be able to convert \( [C/\alpha]A \) to \( C \) and vice versa so as to create or analyze a value of type \( C \). Thus, under the iso-recursive approach, a declaration of a recursive type \( C = \mu \alpha . A \) implicitly introduces two primitive constructs \( \text{fold}_C \) and \( \text{unfold}_C \) specialized for type \( C \). Operationally we may think of \( \text{fold}_C \) and \( \text{unfold}_C \) as behaving like functions of the following types:

\[
\begin{align*}
\text{fold}_C : [C/\alpha]A & \rightarrow C \\
\text{unfold}_C : C & \rightarrow [C/\alpha]A
\end{align*}
\]

As \( \text{fold}_C \) and \( \text{unfold}_C \) are actually not functions but primitive constructs which always require an additional expression as an argument (i.e., we cannot treat \( \text{fold}_C \) as a first-class object), the abstract syntax is extended as follows:

\[
\begin{align*}
\text{expression} & \quad e ::= \cdots \mid \text{fold}_C \ e \mid \text{unfold}_C \ e \\
\text{value} & \quad v ::= \cdots \mid \text{fold}_C \ v
\end{align*}
\]

The typing rules for \( \text{fold}_C \) and \( \text{unfold}_C \) are derived from the operational interpretation of \( \text{fold}_C \) and \( \text{unfold}_C \) given above:

\[
\begin{align*}
C = \mu \alpha . A & \quad \Gamma \vdash e : [C/\alpha]A & \quad \Gamma \vdash C \text{ type} \\
\Gamma \vdash \text{fold}_C \ e : C & \quad \text{Fold} \\
\Gamma \vdash \mu \alpha . A & \quad \Gamma \vdash e : C & \quad \text{Unfold}
\end{align*}
\]

The following reduction rules are based on the eager reduction strategy:

\[
\begin{align*}
\begin{array}{c}
e \mapsto e' \\
\text{Fold}
\end{array} \quad \begin{array}{c}
\text{fold}_C \ e \mapsto \text{fold}_C \ e'
\end{array}
\end{align*}
\]  
\[
\begin{align*}
\begin{array}{c}
e \mapsto e' \\
\text{Unfold}
\end{array} \quad \begin{array}{c}
\text{unfold}_C \ e \mapsto \text{unfold}_C \ e'
\end{array}
\end{align*}
\]

**Exercise 1.1.** Propose reduction rules for the lazy reduction strategy.

## 1.2 Recursive data structures

This section presents a few examples of translating datatype declarations of SML to recursive types. The key idea is two-fold: (1) each datatype declaration in SML implicitly introduces a recursive type;
(2) each data constructor belonging to datatype \( C \) implicitly uses \( \text{fold}_C \) and each pattern match for datatype \( C \) implicitly uses \( \text{unfold}_C \).

Let us begin with a non-recursive datatype which does not have to be translated to a recursive type:

\[
\text{datatype \( \text{bool} = \text{True} \mid \text{False} \)}
\]

Since a value of type \( \text{bool} \) is either \( \text{True} \) or \( \text{False} \), we translate \( \text{bool} \) to a sum type \( \text{unit + unit} \) so as to express the existence of two alternatives; the use of type \( \text{unit} \) indicates that data constructors \( \text{True} \) and \( \text{False} \) require no argument:

\[
\begin{align*}
\text{bool} &= \text{unit + unit} \\
\text{True} &= \text{inl}_\text{unit}(()) \\
\text{False} &= \text{inr}_\text{unit}() \\
\text{if } e \text{ then } e_1 \text{ else } e_2 &= \text{case } e \text{ of } \text{inl } e_1 \mid \text{inr } e_2
\end{align*}
\]

Thus data constructors, which are separated by \( | \) in a datatype declaration, become separated by \( + \) when translated to a type in the simply typed \( \lambda \)-calculus.

Now consider a recursive datatype for natural numbers:

\[
\text{datatype \( \text{nat} = \text{Zero} \mid \text{Succ of nat} \)}
\]

A recursive type for \( \text{nat} \) is the solution to the equation \( \text{nat} \cong \text{unit + nat} \) where the left \( \text{unit} \) corresponds to \( \text{Zero} \) and the right \( \text{nat} \) corresponds to \( \text{Succ} \):

\[
\text{nat} = \mu \alpha. \text{unit + } \alpha
\]

Then both data constructors \( \text{Zero} \) and \( \text{Succ} \) first prepare a value of type \( \text{unit + nat} \) and then “fold” it to create a value of type \( \text{nat} \):

\[
\begin{align*}
\text{Zero} &= \text{fold}_\text{nat} \text{ inl}_\text{nat} () \\
\text{Succ } e &= \text{fold}_\text{nat} \text{ inr}_\text{unit } e
\end{align*}
\]

A pattern match for datatype \( \text{nat} \) works in the opposite way: it first “unfolds” a value of type \( \text{nat} \) to obtain a value of type \( \text{unit + nat} \) which is then analyzed by a case expression:

\[
\text{case } e \text{ of Zero } \Rightarrow e_1 \mid \text{Succ } x \Rightarrow e_2 = \text{case } \text{unfold}_\text{nat } e \text{ of } \text{inl } e_1 \mid \text{inr } x.e_2
\]

Similarly a recursive datatype for lists of natural numbers is translated as follows:

\[
\begin{align*}
\text{datatype } \text{nlist} &= \text{Nil} \mid \text{Cons of nat \times nlist} \\
\text{nlist} &= \mu \alpha. \text{unit + (nat \times nlist)} \\
\text{Nil} &= \text{fold}_\text{nlist} \text{ inl}_\text{nlist} () \\
\text{Cons } e &= \text{fold}_\text{nlist} \text{ inr}_\text{unit } e \\
\text{case } e \text{ of } \text{Nil } \Rightarrow e_1 \mid \text{Cons } x \Rightarrow e_2 &= \text{case } \text{unfold}_\text{nlist } e \text{ of } \text{inl } e_1 \mid \text{inr } x.e_2
\end{align*}
\]

As an example of a recursive type that does not use a sum type, let us consider a datatype for streams of natural numbers:

\[
\begin{align*}
\text{datatype } \text{nstream} &= \text{Nstream of unit \rightarrow nat \times nstream} \\
\text{nstream} &= \mu \alpha. \text{unit \rightarrow nat \times } \alpha
\end{align*}
\]

When “unfolded,” a value of type \( \text{nstream} \) yields a function of type \( \text{unit \rightarrow nat \times nstream} \) which returns a natural number and another stream. For example, the following \( \lambda \)-abstraction has type \( \text{nat \rightarrow nat \times nstream} \):

\[
\lambda s: \text{nstream}. \text{unfold}_\text{nstream} s ()
\]

The following function, of type \( \text{nat \rightarrow nstream} \), returns a stream of natural numbers beginning with its argument:

\[
\lambda n: \text{nat}. (\fix f: \text{nat \rightarrow nstream}. \lambda x: \text{nat}. \text{fold}_\text{nstream} \lambda y: \text{unit}. (x, f (\text{Succ } x))) n
\]

**Exercise 1.2.** Why do we need no reduction rule for \( \text{fold}_C \)?
1.3 Typing the untyped λ-calculus

A further application of recursive types is a translation of the untyped λ-calculus to the simply typed λ-calculus augmented with recursive types. Specifically we wish to translate the untyped λ-calculus to the simply typed λ-calculus with the following definition:

\[
\begin{align*}
\text{type} & \quad A & ::= & \; A \to A \mid \alpha \mid \mu \alpha. A \\
\text{expression} & \quad e & ::= & \; x \mid \lambda x : A. e \mid e \; e \mid \text{fold}_A \; e \mid \text{unfold}_A \; e
\end{align*}
\]

Note that unlike the pure simply typed λ-calculus, the definition of types does not include base types.

We translate an expression \(e\) in the untyped λ-calculus to an expression \(e^o\) in the simply typed λ-calculus. We treat all expressions in the untyped λ-calculus alike by assigning a unique type \(\Omega\) (i.e., \(e^o\) is to have type \(\Omega\)). Then the key to the translation is to find such a unique type \(\Omega\).

It is not difficult to find such a type \(\Omega\) when recursive types are available. If every expression is assigned type \(\Omega\), we may think that \(\lambda x. e\) is assigned type \(\Omega \to \Omega\) as well as type \(\Omega\). Or, in order for \(e_1 \; e_2\) to be assigned type \(\Omega\), \(e_1\) must be assigned not only type \(\Omega\) but also type \(\Omega \to \Omega\) because \(e_2\) is assigned type \(\Omega\). Thus \(\Omega\) must be identified with \(\Omega \to \Omega\) (i.e., \(\Omega \cong \Omega \to \Omega\)) and is defined as follows:

\[
\Omega \; = \; \mu \alpha. \alpha \to \alpha
\]

Then expressions in the untyped λ-calculus are translated as follows:

\[
\begin{align*}
x^o & \; = \; x \\
(\lambda x. \; e)^o & \; = \; \text{fold}_\Omega \; \lambda x : \Omega. \; e^o \\
(e_1 \; e_2)^o & \; = \; (\text{unfold}_\Omega \; e_1^o) \; e_2^o
\end{align*}
\]

**Proposition 1.3.** \(\vdash e^o : \Omega\) holds for any expression \(e\) in the untyped λ-calculus.

**Proposition 1.4.** If \(e \; \leftrightarrow \; e'\), then \(e^o \; \leftrightarrow^* \; e'^o\).

In Proposition 1.4, extra reduction steps in \(e^o \; \leftrightarrow^* \; e'^o\) are due to applications of the rule \text{Unfold}^2.

An interesting consequence of the translation is that despite the absence of the fixed point construct, the reduction of an expression in the simply typed λ-calculus with recursive types may not terminate! For example, the reduction of \(((\lambda x. \; x) \; (\lambda x. \; x))\)^o does not terminate because \((\lambda x. \; x) \; (\lambda x. \; x)\) reduces to itself. In fact, we can even write recursive functions — all we have to do is to translate the fixed point combinator \text{fix} (see Section ??)!

1.4 Exercises

**Exercise 1.5.** Consider the simply typed λ-calculus augmented with recursive types. We use a function type \(A \to B\) for non-recursive functions from type \(A\) to type \(B\). Now let us introduce another function type \(A \Rightarrow B\) for recursive functions from type \(A\) to type \(B\). Define \(A \Rightarrow B\) in terms of ordinary function types and recursive types.