Chapter 1

Simply typed \(\lambda\)-calculus

This chapter presents the *simply typed \(\lambda\)-calculus*, an extension of the \(\lambda\)-calculus with *types*. Since the \(\lambda\)-calculus in the previous chapter does not use types, we refer to it as the *untyped \(\lambda\)-calculus* so that we can differentiate it from the simply typed \(\lambda\)-calculus.

Unlike the untyped \(\lambda\)-calculus in which base types (such as boolean and integers) are simulated with \(\lambda\)-abstractions, the simply typed \(\lambda\)-calculus assumes a fixed set of base types with primitive constructs. For example, we may choose to include a base type `bool` with boolean constants `true` and `false` and a conditional construct `if \(e\) then \(e_1\) else \(e_2\)`. Thus the simply typed \(\lambda\)-calculus may be thought of as not just a core calculus for investigating the expressive power but indeed a subset of a functional language. Then any expression in the simply typed \(\lambda\)-calculus can be literally translated in a functional language such as SML.

As with the untyped \(\lambda\)-calculus, we first formulate the abstract syntax and operational semantics of the simply typed \(\lambda\)-calculus. The difference in the operational semantics is nominal because types play no role in reducing expressions. A major change arises from the introduction of a *type system*, a collection of judgments and inference rules for assigning types to expressions. The type assigned to an expression determines the form of the value to which it evaluates. For example, an expression of type `bool` may evaluate to either `true` or `false`, but nothing else.

The focus of the present chapter is on *type safety*, the most basic property of a type system that an expression with a valid type, or a *well-typed* expression, cannot go wrong at runtime. Since an expression is assigned a type at compile time and type safety ensures that a well-typed expression is well-behaved at runtime, we do not need the trial and error method (of running a program to locate the source of bugs in it) in order to detect fatal bugs such as adding memory addresses, subtracting an integer from a string, using an integer as a destination address in a function invocation, and so forth. Since it is often these simple (and stupid) bugs that cause considerable delay in software development, type safety offers a huge advantage over those programming languages without type systems or with type systems that fail to support type safety. Type safety is also the reason behind the phenomenon that programs that successfully compile run correctly in many cases.

Every extension to the simply typed \(\lambda\)-calculus discussed in this course will preserve type safety. We definitely do not want to squander our time developing a programming language as uncivilized as C!

### 1.1 Abstract syntax

The abstract syntax for the simply typed \(\lambda\)-calculus is given as follows:

<table>
<thead>
<tr>
<th>Type</th>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>A</code></td>
<td></td>
<td>`P</td>
</tr>
<tr>
<td><code>P</code></td>
<td></td>
<td><code>bool</code></td>
</tr>
<tr>
<td><code>e</code></td>
<td></td>
<td>`x</td>
</tr>
<tr>
<td><code>v</code></td>
<td></td>
<td>`\lambda x: A. e</td>
</tr>
</tbody>
</table>
A type is either a base type \( P \) or a function type \( A \rightarrow A' \). A base type is a type whose primitive constructs are given as part of the definition. Here we use a boolean type \( \text{bool} \) as a base type with which three primitive constructs are associated: boolean constants \( \text{true} \) and \( \text{false} \) and a conditional construct \( e \text{ if } e_1 \text{ else } e_2 \). A function type \( A \rightarrow A' \) describes those functions taking an argument of type \( A \) and returning a result of type \( A' \). We use metavariables \( A, B, C \) for types.

It is important that the simply typed \( \lambda \)-calculus does not stipulate specific base types. In other words, the simply typed \( \lambda \)-calculus is just a framework for functional languages whose type system is extensible with additional base types. For example, the definition above considers \( \text{bool} \) as the only base type, but it should also be clear how to extend the definition with another base type (e.g., an integer type \( \text{int} \) with integer constants and arithmetic operators). On the other hand, the simply typed \( \lambda \)-calculus must have at least one base type. Otherwise the set \( P \) of base types is empty, which in turn makes the set \( A \) of types empty. Then we would never be able to create an expression with a valid type!

As in the untyped \( \lambda \)-calculus, expressions include variables, \( \lambda \)-abstractions or functions, and applications. A \( \lambda \)-abstraction \( \lambda x: A. e \) now explicitly specifies the type \( A \) of its formal argument \( x \). If \( \lambda x: A. e \) is applied to an expression of a different type \( A' \) (i.e., \( A \neq A' \)), the application does not typecheck and thus has no type, as will be seen in Section 1.3. We say that variable \( x \) is bound to type \( A \) in a \( \lambda \)-abstraction \( \lambda x: A. e \), or that a \( \lambda \)-abstraction \( \lambda x: A. e \) binds variable \( x \) to type \( A \).

### 1.2 Operational semantics

The development of the operational semantics of the simply typed \( \lambda \)-calculus is analogous to the case for the untyped \( \lambda \)-calculus: we define a mapping \( \text{FV}(e) \) to calculate the set of free variables in \( e \), a capture-avoiding substitution \( [e'/x]e \), and a reduction judgment \( e \rightarrow e' \) with reduction rules. Since the simply typed \( \lambda \)-calculus is different from the untyped \( \lambda \)-calculus except for its use of a type system, its operational semantics reverts to the operational semantics of the untyped \( \lambda \)-calculus if we ignore types in expressions.

A mapping \( \text{FV}(e) \) is defined as follows:

\[
\begin{align*}
\text{FV}(x) &= \{x\} \\
\text{FV}(\lambda x: A. e) &= \text{FV}(e) - \{x\} \\
\text{FV}(e_1 e_2) &= \text{FV}(e_1) \cup \text{FV}(e_2) \\
\text{FV}(\text{true}) &= \emptyset \\
\text{FV}(\text{false}) &= \emptyset \\
\text{FV}(\text{if } e \text{ then } e_1 \text{ else } e_2) &= \text{FV}(e) \cup \text{FV}(e_1) \cup \text{FV}(e_2)
\end{align*}
\]

As in the untyped \( \lambda \)-calculus, we say that an expression is closed if it contains no free variables.

A capture-avoiding substitution \( [e'/x]e \) is defined as follows:

\[
\begin{align*}
[e'/x]x &= e' \\
[e'/x]y &= y \\
[e'/x](\lambda x: A. e) &= \lambda x: A. e \\
[e'/x](\lambda y: A. e) &= \lambda y: A. [e'/x]e \\
[e'/x](e_1 e_2) &= [e'/x]e_1 [e'/x]e_2 \\
[e'/x]\text{true} &= \text{true} \\
[e'/x]\text{false} &= \text{false} \\
[e'/x]\text{if } e \text{ then } e_1 \text{ else } e_2 &= \text{if } [e'/x]e \text{ then } [e'/x]e_1 \text{ else } [e'/x]e_2
\end{align*}
\]

When a variable capture occurs in \( [e'/x]\lambda y: A. e \), we rename the bound variable \( y \) using the \( \alpha \)-equivalence relation \( \equiv_\alpha \). We omit the definition of \( \equiv_\alpha \) because it requires no further consideration than the definition given in Chapter ??.

As with the untyped \( \lambda \)-calculus, different reduction strategies yield different reduction rules for the reduction judgment \( e \rightarrow e' \). We choose the call-by-value strategy which lends itself well to extend-
ing the simply typed $\lambda$-calculus with computational effects such as mutable references, exceptions, and continuations (to be discussed in subsequent chapters). Thus we use the following reduction rules:

\[
\begin{array}{c}
\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \quad \text{Lam} \\
\frac{e_2 \rightarrow e'_2}{(\lambda x : A. e) e_2 \rightarrow (\lambda x : A. e) e'_2} \quad \text{Arg} \\
\frac{e \rightarrow e'}{\text{if } e \text{ then } e_1 \text{ else } e_2 \rightarrow \text{if } e' \text{ then } e_1 \text{ else } e_2} \quad \text{If}
\end{array}
\]

\[
\begin{array}{c}
\frac{e \rightarrow e'}{\text{if true then } e \rightarrow e_1} \quad \text{If}_\text{true} \\
\frac{e \rightarrow e'}{\text{if false then } e \rightarrow e_2} \quad \text{If}_\text{false}
\end{array}
\]

The rules $\text{Lam}$, $\text{Arg}$, and $\text{App}$ are exactly the same as in the untyped $\lambda$-calculus except that we use a $\lambda$-abstraction of the form $\lambda x : A. e$. (To implement the call-by-name strategy, we remove the rule $\text{Arg}$ and rewrite the rule $\text{App}$ as $\frac{(\lambda x : A. e) e' \rightarrow [e'/x]e}{\text{App}}$. ) The three rules $\text{If}_\text{true}$, and $\text{If}_\text{false}$ combined together specify how to reduce a conditional construct if $e$ then $e_1$ else $e_2$:

- We reduce $e$ to either true or false.
- If $e$ reduces to true, we choose the then branch and begin to reduce $e_1$.
- If $e$ reduces to false, we choose the else branch and begin to reduce $e_2$.

As before, we write $\rightarrow^*$ for the reflexive and transitive closure of $\rightarrow$. We say that $e$ evaluates to $v$ if $e \rightarrow^* v$ holds.

### 1.3 Type system

The goal of this section is to develop a system of inference rules for assigning types to expressions in the simply typed $\lambda$-calculus. We use a judgment called a typing judgment, and refer to inference rules deducing a typing judgment as typing rules. The resultant system is called the type system of the simply typed $\lambda$-calculus.

To figure out the right form for the typing judgment, let us consider an identity function $id = \lambda x : A. x$. Intuitively id has a function type $A \rightarrow A$ because it takes an argument of type $A$ and returns a result of the same type. Then how do we determine, or “infer,” the type of id? Since id is a $\lambda$-abstraction with an argument of type $A$, all we need is the type of its body. It is easy to see, however, that its body cannot be considered in isolation: without any assumption on the type of its argument $x$, we cannot infer the type of its body $x$.

The example of id suggests that it is inevitable to use assumptions on types of variables in typing judgments. Thus we are led to introduce a typing context to denote an unordered set of assumptions on types of variables; we use a type binding $x : A$ to mean that variable $x$ assumes type $A$:

\[
\text{typing context} \quad \Gamma ::= \cdot \mid \Gamma, x : A
\]

- $\cdot$ denotes an empty typing context and is our notation for an empty set $\emptyset$.
- $\Gamma, x : A$ augments $\Gamma$ with a type binding $x : A$ and is our notation for $\Gamma \cup \{ x : A \}$. We abbreviate $\cdot, x : A$ as $x : A$ to denote a singleton typing context $\{ x : A \}$.
- We use the notation for typing contexts in a flexible way. For example, $\Gamma, x : A, \Gamma'$ denotes $\Gamma \cup \{ x : A \} \cup \Gamma'$, and $\Gamma, \Gamma'$ denotes $\Gamma \cup \Gamma'$.

For the sake of simplicity, we assume that variables in a typing context are all distinct. That is, $\Gamma, x : A$ is not defined if $\Gamma$ contains another type binding of the form $x : A'$, or simply if $x : A' \in \Gamma$.

The type system uses the following form of typing judgment:

\[
\Gamma \vdash e : A \quad \iff \quad \text{expression } e \text{ has type } A \text{ under typing context } \Gamma
\]
\[ \Gamma \vdash e : A \] means that if we use each type binding \( x : A \) in \( \Gamma \) as an assumption, we can show that expression \( e \) has type \( A \). An easy way to understand the role of \( \Gamma \) is by thinking of it as a set of type bindings for free variables in \( e \), although \( \Gamma \) may also contain type bindings for those variables not found in \( e \). For example, a closed expression \( e \) of type \( A \) needs a typing judgment \( \Gamma \vdash e : A \) with an empty typing context (because it contains no free variables), whereas an expression \( e' \) with a free variable \( x \) needs a typing judgment \( \Gamma \vdash e' : A' \) where \( \Gamma \) contains at least a type binding \( x : B \) for some type \( B \).

With the above interpretation of typing judgments, we can now explain the typing rules for the simply typed \( \lambda \)-calculus:

\[
\begin{array}{c}
\text{Var} & \quad \frac{x : A \in \Gamma}{\Gamma \vdash x : A} \\
\text{If} & \quad \frac{\Gamma \vdash e : A \to B \quad \Gamma \vdash e' : A}{\Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : A} \\
\text{True} & \quad \frac{\Gamma \vdash \text{true} : \text{bool}}{}
\end{array}
\]

- The rule \text{Var} means that a type binding in a typing context is an assumption. Alternatively we may rewrite the rule as follows:
  \[ \frac{\Gamma, x : A, \Gamma' \vdash x : A}{\text{Var}} \]

- The rule \( \rightarrow \text{l} \) says that if \( e \) has type \( B \) under the assumption that \( x \) has type \( A \), then \( \lambda x : A. e \) has type \( A \to B \). If we read the rule \( \rightarrow \text{l} \) from the premise to the conclusion (i.e., top-down), we “introduce” a function type \( A \to B \) from the judgment in the premise, which is the reason why it is called the “\( \to \) Introduction rule.” Note that if \( \Gamma \) already contains a type binding for variable \( x \) (i.e., \( x : A' \in \Gamma \)), we rename \( x \) to a fresh variable by \( \alpha \)-conversion. Hence we may assume without loss of generality that variable clashes never occur in the rule \( \rightarrow \text{l} \).

- The rule \( \rightarrow \text{E} \) says that if \( e \) has type \( A \to B \) and \( e' \) has type \( A \), then \( e \, e' \) has type \( B \). If we read the rule \( \rightarrow \text{E} \) from the premise to the conclusion, we “eliminate” a function type \( A \to B \) to produce an expression of a smaller type \( B \), which is the reason why it is called the “\( \to \) Elimination rule.”

- The rules \text{True} and \text{False} assign base type \text{bool} to boolean constants \text{true} and \text{false}. Note that typing context \( \Gamma \) is not used because there is no free variable in \text{true} and \text{false}.

- The rule \text{If} says that if \( e \) has type \text{bool} and both \( e_1 \) and \( e_2 \) have the same type \( A \), then \( \text{if } e \text{ then } e_1 \text{ else } e_2 \) has type \( A \).

A derivation tree for a typing judgment is called a typing derivation. Here are a few examples of valid typing derivations. The first example infers the type of an identity function (where we use no premise in the rule \text{Var}):

\[
\frac{}{\frac{}{\text{Var} \quad \frac{\Gamma, x : A \vdash x : A}{\Gamma \vdash \lambda x : A. x : A \to A}}}
\]

Since \( \lambda x : A. x \) is closed, we may use an empty typing context \( \cdot \) for \( \Gamma \):

\[
\frac{}{\frac{\cdot \vdash x : A \vdash x : A}{\text{Var} \quad \frac{\cdot \vdash \lambda x : A. x : A \to A}}}
\]

In the second example below, we abbreviate \( \Gamma, x : \text{bool}, y_1 : A, y_2 : A \) as \( \Gamma' \). Note also that \( \text{bool} \to A \to A \to A \)

---

\(^1\) A typing judgment \( \Gamma \vdash e : A \) is an example of a hypothetical judgment which deduces a “judgment” \( e : A \) using each “judgment” \( x : A \) in \( \Gamma \) as a hypothesis. From this point of view, the turnstile symbol \( \vdash \) is just a syntactic device which plays no semantic role at all. Although the notion of hypothetical judgment is of great significance in the study of logic, I do not find it particularly useful in helping students to understand the type system of the simply typed \( \lambda \)-calculus.
is equivalent to \( (A \to (A \to A)) \) because \( \to \) is right-associative:

\[
\begin{align*}
  x : \text{bool} & \in \Gamma' \quad \frac{}{\Gamma' \vdash x : \text{bool}} \\
  y_1 : A & \in \Gamma' \quad \frac{}{\Gamma' \vdash y_1 : A} \\
  y_2 : A & \in \Gamma' \quad \frac{}{\Gamma' \vdash y_2 : A} \\
  f : A \to B & \in \Gamma' \quad \frac{}{\Gamma' \vdash f : A \to B} \\
  x : A & \in \Gamma' \quad \frac{}{\Gamma' \vdash x : A} \\
  f \in \Gamma' \quad \frac{\Gamma, x : A, y_1 : A, y_2 : A \vdash \text{if } y_1 \text{ then } y_1 \text{ else } y_2 : A}{\Gamma \vdash \text{if } y_1 \text{ then } y_1 \text{ else } y_2 : A \to A \\l} \\
  y_1 : A & \in \Gamma' \quad \frac{}{\Gamma \vdash \lambda y_1 : A. y_1 : A} \\
  y_2 : A & \in \Gamma' \quad \frac{}{\Gamma \vdash \lambda y_2 : A. y_2 : A} \\
  g : B \to C & \in \Gamma' \quad \frac{}{\Gamma \vdash g : B \to C} \\
  f \in \Gamma' \quad \frac{\Gamma, x : A \vdash g (f x) : C}{\Gamma \vdash g (f x) : C} \\
  f \in \Gamma' \quad \frac{\Gamma, x : A \vdash g (f x) : C}{\Gamma \vdash \lambda x : A. g (f x) : A \to C} \\
  \lambda f : A \to B & \in \Gamma' \quad \frac{}{\Gamma \vdash \lambda f : A \to B} \\
  \lambda g : B \to C & \in \Gamma' \quad \frac{}{\Gamma \vdash \lambda g : B \to C} \\
  \lambda x : A. g (f x) & \in \Gamma' \quad \frac{}{\Gamma \vdash \lambda x : A. g (f x) : (A \to B) \to (B \to C) \to (A \to C)} \\
\end{align*}
\]

The third example infers the type of a function composing two functions \( f \) and \( g \) where we abbreviate \( \Gamma, f : A \to B, g : B \to C, x : A \) as \( \Gamma' \):

\[
\begin{align*}
  g : B \to C & \in \Gamma' \quad \frac{}{\Gamma' \vdash g : B \to C} \\
  f : A \to B & \in \Gamma' \quad \frac{}{\Gamma' \vdash f : A \to B} \\
  x : A & \in \Gamma' \quad \frac{}{\Gamma' \vdash x : A} \\
  f \in \Gamma' \quad \frac{\Gamma, x : A, y : B, x : A, y : B, \Gamma \vdash y : B \to C \text{ and } x : A, y : B, \Gamma \vdash x : A}{\Gamma \vdash y : B \to C} \\
  \lambda f : A \to B, g : B \to C, x : A \vdash g (f x) : C \\
  \lambda f : A \to B, g : B \to C & \in \Gamma' \quad \frac{}{\Gamma \vdash \lambda f : A \to B, g : B \to C} \\
  \lambda x : A. g (f x) & \in \Gamma' \quad \frac{}{\Gamma \vdash \lambda x : A. g (f x) : (A \to B) \to (B \to C) \to (A \to C)} \\
\end{align*}
\]

We close this section by proving two properties of typing judgments: permutation and weakening. The permutation property reflects the assumption that a typing context \( \Gamma \) is an unordered set, which means that two typing contexts are identified up to permutation. For example, \( \Gamma, x : A, y : B \) is identified with \( \Gamma, y : B, x : A, \Gamma, y : B \), with \( x : A, y : B, \Gamma \), and so on. The weakening property says that if we can prove that expression \( e \) has type \( A \) under typing context \( \Gamma \), we can also prove it under another typing context \( \Gamma' \) augmenting \( \Gamma \) with a new type binding \( x : A \) (because we can just ignore the new type binding). These properties are called structural properties of typing judgments because they deal with the structure of typing judgments rather than their derivations\(^2\).

**Proposition 1.1 (Permutation).** If \( \Gamma \vdash e : A \) and \( \Gamma' \) is a permutation of \( \Gamma \), then \( \Gamma' \vdash e : A \).

**Proof.** By rule induction on the judgment \( \Gamma \vdash e : A \).

**Proposition 1.2 (Weakening).** If \( \Gamma \vdash e : C \), then \( \Gamma, x : A \vdash e : C \).

**Proof.** By rule induction on the judgment \( \Gamma \vdash e : C \). We show three cases. The remaining cases are similar to the case for the rule \( \to \text{E} \).

**Case**

\[
\begin{align*}
  y : C & \in \Gamma' \quad \frac{}{\Gamma' \vdash y : C} \\
  y : C & \in \Gamma \quad \frac{}{\Gamma \vdash y : C} \\
  x : A & \in \Gamma \quad \frac{}{\Gamma, x : A \vdash y : C} \\
  \Gamma & \vdash e : \text{Var} \\
\end{align*}
\]

**Case**

\[
\begin{align*}
  \lambda y : C_1, e' : C_2 & \in \Gamma \quad \frac{\Gamma, y : C_1 \vdash e' : C_2}{\Gamma \vdash \lambda y : C_1, e' : C_2} \\
  \lambda y : C_1, e' : C_2 & \in \Gamma' \quad \frac{}{\Gamma' \vdash \lambda y : C_1, e' : C_2} \\
  \lambda y : C_1, e' : C_2 & \in \Gamma' \quad \frac{}{\Gamma' \vdash \lambda y : C_1, e' : C_2} \\
  \lambda y : C_1, e' : C_2 & \in \Gamma' \quad \frac{}{\Gamma' \vdash \lambda y : C_1, e' : C_2} \\
\end{align*}
\]

This is the case where \( \Gamma \vdash e : C \) is proven by applying the rule \( \text{E} \). In other words, the last inference rule applied in the proof of \( \Gamma \vdash e : C \) is the rule \( \text{E} \). Then \( e \) must have the form \( \lambda y : C_1, e' \) for some type \( C = C_1 \to C_2 \); otherwise the rule \( \text{E} \) cannot be applied. Then the premise is uniquely determined as \( \Gamma, y : C_1 \vdash e' : C_2 \).

![There is no strict rule on whether a property should be called a theorem or a proposition, although a general rule of thumb is that a property relatively easy to prove is called a proposition. A property that is of great significance is usually called a theorem.](image)

\(^2\) There is no strict rule on whether a property should be called a theorem or a proposition, although a general rule of thumb is that a property relatively easy to prove is called a proposition. A property that is of great significance is usually called a theorem. (For example, it is Fermat’s last theorem rather than Fermat’s last proposition.) A lemma is a property proven for facilitating proofs of other theorems, propositions, or lemmas.
Case \( \Gamma 
vdash e_1 : B \rightarrow C \quad \Gamma 
vdash e_2 : B \rightarrow C \quad \rightarrow E \quad \text{where } e = e_1 e_2; \)

This is the case where \( \Gamma 
vdash e : C \) is proven by applying the rule \( \rightarrow E \). Then \( e \) must have the form \( e_1 e_2 \) and the two premises are uniquely determined for some type \( B \); otherwise the rule \( \rightarrow E \) cannot be applied.

\[
\begin{align*}
\Gamma, x : A &\vdash e_1 : B \rightarrow C \\
&\text{by induction hypothesis on } \Gamma 
vdash e_1 : B \rightarrow C \\
\Gamma, x : A &\vdash e_2 : B \\
&\text{by induction hypothesis on } \Gamma 
vdash e_2 : B \\
\Gamma, x : A &\vdash e_1 e_2 : C \\
&\text{by the rule } \rightarrow E
\end{align*}
\]

As typing contexts are always assumed to be unordered sets, we implicitly use the permutation property in proofs. For example, we may deduce \( \Gamma, y : C_1, e' : C_1 \rightarrow C_2 \) directly from \( \Gamma, y : C_1, x : A 
vdash e' : C_2 \) without an intermediate step of permutating \( \Gamma, y : C_1, x : A \) into \( \Gamma, x : A, y : C_1 \).

### 1.4 Type safety

In order to determine properties of expressions, we have developed two systems for the simply typed \( \lambda \)-calculus: operational semantics and type system. The operational semantics enables us to find out dynamic properties, namely values, associated with expressions. Values are dynamic properties in the sense that they can be determined only at runtime in general. For this reason, an operational semantics is also called a *dynamic semantics*. In contrast, the type system enables us to find out static properties, namely types, of expressions. Types are static properties in the sense that they are determined at compile time and remain “static” at runtime. For this reason, a type system is also called a *static semantics*.

We have developed the type system independently of the operational semantics. Therefore there remains a possibility that it does not respect the operational semantics, whether intentionally or unintentionally. For example, it may assign different types to two expressions \( e \) and \( e' \) such that \( e \mapsto e' \), which is unnatural because we do not anticipate a change in type when an expression reduces to another expression. Or it may assign a valid type to a nonsensical expression, which is also unnatural because we do not anticipate a change in type when an expression reduces to another type.

Type safety consists of two theorems: *progress* and *type preservation*. The progress theorem states that a (closed) well-typed expression is not stuck: either it is a value or it reduces to another expression:

**Theorem 1.3 (Progress).** If \( \cdot \vdash e : A \) for some type \( A \), then either \( e \) is a value or there exists \( e' \) such that \( e \mapsto e' \).

The type preservation theorem states that when a well-typed expression reduces, the resultant expression is also well-typed and has the same type; type preservation is also called *subject reduction*:

**Theorem 1.4 (Type preservation).** If \( \Gamma 
vdash e : A \) and \( e \mapsto e' \), then \( \Gamma 
vdash e' : A \).

Note that the progress theorem assumes an empty typing context (hence a *closed* well-typed expression \( e \)) whereas the type preservation theorem does not. It actually makes sense if we consider whether a reduction judgment \( e \mapsto e' \) is part of the conclusion or is given as an assumption. In the case of the progress theorem, we are interested in whether \( e \) reduces to another expression or not, provided that it is well-typed. Therefore we use an empty typing context to disallow free variables in \( e \) which may make its reduction impossible. If we allowed any typing context \( \Gamma \), the progress theorem would be downright false, as evidenced by a simple counterexample \( e = x \) which is not a value and is irreducible. In the case of the type preservation theorem, we begin with an assumption \( e \mapsto e' \). Then there is no reason not to allow free variables in \( e \) because we already know that it reduces to another expression \( e' \). Thus we use a metavariable \( \Gamma \) (ranging over all typing contexts) instead of an empty typing context.

Combined together, the two theorems guarantee that a (closed) well-typed expression never reduces to a stuck expression: either it is a value or it reduces to another well-typed expressions. Consider a well-typed expression \( e \) such that \( \cdot \vdash e : A \) for some type \( A \). If \( e \) is already a value, there is no need to reduce it (and hence it is not stuck). If not, the progress theorem ensures that there exists an expression \( e' \) such that \( e \mapsto e' \), which is also a well-typed expression of the same type \( A \) by the type preservation theorem.
Below we prove the two theorems using rule induction. It turns out that a direct proof attempt by rule induction fails, and thus we need a couple of lemmas. These lemmas (canonical forms and substitution) are so prevalent in programming language theory that their names are worth memorizing.

### 1.4.1 Proof of progress

The proof of Theorem 1.3 is relatively straightforward: the theorem is written in the form "If $J$ holds, then $P(J)$ holds," and we apply rule induction to the judgment $J$, which is a typing judgment $\vdash e : A$. So we begin with an assumption $\vdash e : A$. If $e$ happens to be a value, then $P(J)$ holds trivially because the judgment "$e$ is a value" holds. Thus we make a stronger assumption $\vdash e : A$ with $e$ not being a value. Then we analyze the structure of the proof of $\vdash e : A$, which gives three cases to consider:

1. $e_1$ is a value.
2. $e_1$ is not a value and reduces to another expression $e'_1$, i.e., $e_1 \mapsto e'_1$.

If the second possibility is the case, we have found an expression to which $e_1$ reduces, namely $e_1 \mapsto e'_1$.

Now what if the first possibility is the case? Since $e_1$ has type $A \to B$, it is likely to be a $\lambda$-abstraction, in which case the induction hypothesis on the second premise $\vdash e_2 : A$ opens another two possibilities and we use either the rule $Arg$ or the rule $App$ to show the progress property. Unfortunately we do not have a formal proof that $e_1$ is indeed a $\lambda$-abstraction; we know only that $e_1$ has type $A \to B$ under an empty typing context. Our instinct, however, says that $e_1$ must be a $\lambda$-abstraction because it has type $A \to B$. The following lemma formalizes our instinct on the correct, or “canonical,” form of a well-typed value:

**Lemma 1.5 (Canonical forms).**

If $v$ is a value of type bool, then $v$ is either true or false.

If $v$ is a value of type $A \to B$, then $v$ is a $\lambda$-abstraction $\lambda x : A. e$.

**Proof.** By case analysis of $v$. (Not every proof uses rule induction!)

Suppose that $v$ is a value of type bool. The only typing rules that assign a boolean type to a given value are True and False. Therefore $v$ is a boolean constant true or false. Note that the rules $Var$, $\lnot E$, and $If$ may assign a boolean type, but never to a value.

Suppose that $v$ is a value of type $A \to B$. The only typing rule that assigns a function type to a given value is $\lnot E$. Therefore $v$ must be a $\lambda$-abstraction of the form $\lambda x : A. e$ (which binds variable $x$ to type $A$). Note that the rules $Var$, $\lnot E$, and $If$ may assign a function type, but not to a value.

Now we are ready to prove the progress theorem:

**Proof of Theorem 1.3** By rule induction on the judgment $\vdash e : A$.

If $e$ is already a value, we need no further consideration. Therefore we assume that $e$ is not a value. Then there are three cases to consider.

Case $\vdash x : A \in \cdot$ Var where $e = x$: impossible from $x : A \notin \cdot$.

Case $\vdash e_1 : B \to A \vdash e_2 : B$ $\vdash e_1 \; e_2 : A \to E$ where $e = e_1 \; e_2$.  

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$e_1$ is a value or there exists $e'_1$ such that $e_1 \mapsto e'_1$ by induction hypothesis on $\cdot \vdash e_1 : B \rightarrow A$

$e_2$ is a value or there exists $e'_2$ such that $e_2 \mapsto e'_2$ by induction hypothesis on $\vdash e_2 : B$

Subcase: $e_1$ is a value and $e_2$ is a value

$e_1 = \lambda x : B. e'_1$

$e_2 = v_2$

$e_1 \ e_2 \mapsto [v_2/x]e'_1$

We let $e' = [v_2/x]e'_1$.

Subcase: $e_1$ is a value and there exists $e'_2$ such that $e_2 \mapsto e'_2$

$e_1 = \lambda x : B. e'_1$

$e_1 \ e_2 \mapsto (\lambda x : B. e'_1) \ e'_2$

We let $e' = (\lambda x : B. e'_1) \ e'_2$.

Subcase: there exists $e'_1$ such that $e_1 \mapsto e'_1$

$e_1 \ e_2 \mapsto e'_1 \ e_2$

We let $e' = e'_1 \ e_2$.

Case

\[
\begin{array}{ll}
\text{If } & \text{where } e = \text{if } e_b \text{ then } e_1 \text{ else } e_2;
\end{array}
\]

$e_b$ is either a value or there exists $e'_b$ such that $e_b \mapsto e'_b$ by induction hypothesis on $\vdash e_b : \text{bool}$

Subcase: $e_b$ is a value

$e_b$ is either true or false by Lemma 1.5

if $e_b$ then $e_1$ else $e_2$ $\mapsto$ $e_1$ or if $e_b$ then $e_1$ else $e_2$ $\mapsto$ $e_2$

We let $e' = e_1$ or $e' = e_2$.

Subcase: there exists $e'_b$ such that $e_b \mapsto e'_b$

if $e_b$ then $e_1$ else $e_2$ $\mapsto$ if $e'_b$ then $e_1$ else $e_2$

We let $e' = e'_b$ if $e'_b$ then $e_1$ else $e_2$.

\[\square\]

1.4.2 Proof of type preservation

The proof of Theorem 1.4 is not as straightforward as the proof of Theorem 1.3 because the If part in the theorem contains two judgments: $\Gamma \vdash e : A$ and $e \mapsto e'$. (We have seen a similar case in the proof of Lemma 1.5.) Therefore we need to decide to which judgment of $\Gamma \vdash e : A$ and $e \mapsto e'$ we apply rule induction. It turns out that the type preservation theorem is a special case in which we may apply rule induction to either judgment!

Suppose that we choose to apply rule induction to $e \mapsto e'$. Since there are six reduction rules, we need to consider (at least) six cases. The question now is: which case do we do consider first?

As a general rule of thumb, if you are proving a property that is expected to hold, the most difficult case should be the first to consider. The rationale is that eventually you have to consider the most difficult case anyway, and by considering it at an early stage of the proof, you may find a flaw in the system or identify auxiliary lemmas required for the proof. Even if you discover a flaw in the system from the analysis of the most difficult case, you at least avoid considering easy cases more than once. Conversely, if you are trying to locate flaws in the system by proving a property that is not expected to hold, the easiest case should be the first to consider. The rationale is that the cheapest way to locate a flaw is to consider the easiest case in which the flaw manifests itself (although it is not as convincing as the previous rationale). The most difficult case may not even shed any light on hidden flaws in the system, thereby wasting your efforts to analyze it.

Since we wish to prove the type preservation theorem rather than refute it, we consider the most difficult case of $e \mapsto e'$ first. Intuitively the most difficult case is when $e \mapsto e'$ is proven by applying the rule App, since the substitution in it may transform an application $e$ into a completely different form of expression, for example, a conditional construct. (The rules If\text{true} and If\text{false} are the easiest cases because they have no premise and $e'$ is a subexpression of $e$.)

So let us consider the most difficult case in which $(\lambda x : A. e) \ v \mapsto [v/x]e$ holds. Our goal is to use an
assumption $\Gamma \vdash (\lambda x : A . e) v : C$ to prove $\Gamma \vdash [v/x]e : C$. The typing judgment $\Gamma \vdash (\lambda x : A . e) v : C$ must have the following derivation tree:

$$
\begin{array}{c}
\Gamma, x : A \vdash e : C \\
\Gamma \vdash \lambda x : A . e : A \rightarrow C \rightarrow I \\
\Gamma \vdash \lambda x : A . e v : A \\
\Gamma \vdash (\lambda x : A . e) v : C \\
\end{array}
$$

Therefore our new goal is to use two assumptions $\Gamma, x : A \vdash e : C$ and $\Gamma \vdash v : A$ to prove $\Gamma \vdash [v/x]e : C$. The substitution lemma below generalizes the problem:

**Lemma 1.6 (Substitution).** If $\Gamma \vdash e : A$ and $\Gamma, x : A \vdash e' : C$, then $\Gamma \vdash [e/x]e' : C$.

The substitution lemma is similar to the type preservation theorem in that the $If$ part contains two judgments. Unlike the type preservation theorem, however, we need to take great care in applying rule induction because picking up a wrong judgment makes it impossible to complete the proof!

**Exercise 1.7.** To which judgment do you think we have to apply rule induction in the proof of Lemma 1.6? $\Gamma \vdash e : A$ or $\Gamma, x : A \vdash e' : C$? Why?

The key observation is that $[e/x]e'$ analyzes the structure of $e'$, not $e$. That is, $[e/x]e'$ searches for every occurrence of variable $x$ in $e'$ only to replace it by $e$, and thus does not even need to know the structure of $e$. Thus the right judgment for applying rule induction is $\Gamma, x : A \vdash e' : C$.

**Proof of Lemma 1.6.** By rule induction on the judgment $\Gamma, x : A \vdash e' : C$. Recall that variables in a typing context are assumed to be all distinct. We show four cases. The first two deal with those cases where $e'$ is a variable. The remaining cases are similar to the case for the rule $\rightarrow I$.

**Case** $\Gamma, x : A \vdash e' : C$ where $e' = y$ and $y : C \in \Gamma$:

This is the case where $e'$ is a variable $y$ that is different from $x$. Since $y \neq x$, the premise $y : C \in \Gamma, x : A$ implies the side condition $y : C \in \Gamma$.

$$
\begin{array}{c}
\Gamma \vdash y : C \\
\Gamma \vdash [e/x]y : C \\
\end{array}
$$

**Case** $\Gamma, x : A \vdash x : A$ where $e' = x$ and $C = A$:

This is the case where $e'$ is the variable $x$.

$$
\begin{array}{c}
\Gamma \vdash e : A \\
\Gamma \vdash [e/x]x : A \\
\end{array}
$$

**Case** $\Gamma, x : A \vdash \lambda y : C_1 . e'' : C_2$ where $e' = \lambda y : C_1 . e''$ and $C = C_1 \rightarrow C_2$:

Here we may assume without loss of generality that $y$ is a fresh variable such that $y \notin \text{FV}(e)$ and $y \neq x$. If $y \notin \text{FV}(e)$ or $y = x$, we can always choose a different variable by applying an $\alpha$-conversion to $\lambda y : C_1 . e''$.

$$
\begin{array}{c}
\Gamma, y : C_1 \vdash [e/x]e'' : C_2 \\
\Gamma \vdash \lambda y : C_1 . [e/x]e'' : C_1 \rightarrow C_2 \\
\Gamma \vdash [e/x]\lambda y : C_1 . e'' : C_1 \rightarrow C_2 \\
\end{array}
$$

**Case** $\Gamma, x : A \vdash e_1 : B \rightarrow C$ and $\Gamma, x : A \vdash e_2 : B$ where $e' = e_1 e_2$:

$$
\begin{array}{c}
\Gamma \vdash [e/x]e_1 : B \rightarrow C \\
\Gamma \vdash [e/x]e_2 : B \\
\Gamma \vdash [e/x]e_1 [e/x]e_2 : C \\
\Gamma \vdash [e/x](e_1 e_2) : C \\
\end{array}
$$

At last, we are ready to prove the type preservation theorem. The proof proceeds by rule induction on the judgment $e \mapsto e'$. It exploits the fact that there is only one typing rule for each form of expression.

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For example, the only way to prove $\Gamma \vdash e_1 e_2 : A$ is to apply the rule $-E$. Thus the type system is said to be syntax-directed in that the syntactic form of expression $e$ in a judgment $\Gamma \vdash e : A$ decides, or directs, the rule to be applied. Since the syntax-directedness of the type system decides a unique typing rule $R$ for deducing $\Gamma \vdash e : A$, the premises of the rule $R$ may be assumed to hold whenever $\Gamma \vdash e : A$ holds. For example, $\Gamma \vdash e_1 e_2 : A$ can be proven only by applying the rule $-E$, from which we may conclude that the two premises $\Gamma \vdash e_1 : B \rightarrow A$ and $\Gamma \vdash e_2 : B$ hold for some type $B$. This is called the inversion property which inverts the typing rule so that its conclusion justifies the use of its premises. We state the inversion property as a separate lemma.

**Lemma 1.8 (Inversion).** Suppose $\Gamma \vdash e : C$.

If $e = x$, then $x : C \in \Gamma$.
If $e = \lambda x : A. e'$, then $C = A \rightarrow B$ and $\Gamma, x : A \vdash e' : B$ for some type $B$.
If $e = \text{true}$, then $C = \text{bool}$.
If $e = \text{false}$, then $C = \text{bool}$.
If $e = if \, e_b \, then \, e_1 \, else \, e_2$, then $\Gamma \vdash e_b : \text{bool} \, and \, \Gamma \vdash e_1 : C \, and \, \Gamma \vdash e_2 : C$.

**Proof.** By the syntax-directedness of the type system. A formal proof proceeds by rule induction on the judgment $\Gamma \vdash e : C$.

**Proof of Theorem 1.4.** By rule induction on the judgment $e \rightarrow e'$.

Case $\frac{e_1 \rightarrow e_1'}{e_1 e_2 \rightarrow e_1' e_2}$ Lam

$\Gamma \vdash e_1 e_2 : A$
$\Gamma \vdash e_1 : B \rightarrow A \, and \, \Gamma \vdash e_2 : B$ for some type $B$
$\Gamma \vdash e_1' : B \rightarrow A$
$\Gamma \vdash e_1 e_2 : A$

Case $\frac{\lambda x : B. e_1'}{\lambda x : B. e_1} e_2 \rightarrow (\lambda x : B. e_1') e_2}$ Arg

$\Gamma \vdash \lambda x : B. e_1' e_2 : A$
$\Gamma \vdash \lambda x : B. e_1' : B \rightarrow A \, and \, \Gamma \vdash e_2 : B$
$\Gamma \vdash e_2 : B$
$\Gamma \vdash (\lambda x : B. e_1') e_2 : A$

Case $\frac{(\lambda x : B. e_1') v \rightarrow [v/x] e_1'}{v : A}$ App

$\Gamma \vdash (\lambda x : B. e_1') v : A$
$\Gamma \vdash \lambda x : B. e_1' : B \rightarrow A \, and \, \Gamma \vdash v : B$
$\Gamma, x : B \vdash e_1 : A$
$\Gamma \vdash [v/x] e_1' : A$

Case $\frac{\text{if } e_b \, then \, e_1 \, else \, e_2 \rightarrow e_1'}{\text{if } e_b \, then \, e_1 \, else \, e_2 : A}$ If

$\Gamma \vdash \text{if } e_b \, then \, e_1 \, else \, e_2 : A$
$\Gamma \vdash e_b : \text{bool} \, and \, \Gamma \vdash e_1 : A \, and \, \Gamma \vdash e_2 : A$
$\Gamma \vdash e_b : \text{bool}$
$\Gamma \vdash \text{if } e_b \, then \, e_1 \, else \, e_2 : A$

Case $\frac{\text{if } e_b \, then \, e_1 \, else \, e_2 \rightarrow e_1}{\text{if } e_b \, then \, e_1 \, else \, e_2 : A}$ If

$\Gamma \vdash \text{if } e_b \, then \, e_1 \, else \, e_2 : A$
$\Gamma \vdash \text{true} : \text{bool} \, and \, \Gamma \vdash e_1 : A \, and \, \Gamma \vdash e_2 : A$
$\Gamma \vdash e_1 : A$

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Case $\begin{cases} \text{if false then } e_1 \text{ else } e_2 & \rightarrow e_2 \\ \text{if true} \end{cases}$
(Similar to the case for the rule $If_{true}$)

1.5 Exercises

Exercise 1.9. Prove Theorem 1.4 by rule induction on the judgment $\Gamma \vdash e : A$.

Exercise 1.10. For the simply typed $\lambda$-calculus considered in this chapter, prove the following structural property. The property is called contraction because it enables us to contract $x : A, x : A$ in a typing context to $x : A$.

If $\Gamma, x : A, x : A \vdash e : C$, then $\Gamma, x : A \vdash e : C$.

In your proof, you may assume that a typing context $\Gamma$ is an unordered set. That is, you may identify typing contexts up to permutation. For example, $\Gamma, x : A, y : B$ is identified with $\Gamma, y : B, x : A$. As is already implied by the theorem, however, you may not assume that variables in a typing context are all distinct. A typing context may even contain multiple bindings with different types for the same variable. For example, $\Gamma = \Gamma', x : A, x : B$ is a valid typing context even if $A \neq B$. (In this case, $x$ can have type $A$ or type $B$, and thus typechecking is ambiguous. Still the type system is sound.)