Inductive proofs [40 points]

In this problem, we study a system of strings of matched parentheses. First we define a syntactic category \texttt{paren} for strings of parentheses:

\[
\text{paren} \quad s ::= \epsilon \mid (s) s
\]

\(\epsilon\) stands for the empty string (i.e., \(\epsilon s = s = s \epsilon\)). \texttt{paren} specifies a language of strings of parentheses with no constraint on the use of parentheses.

To identify strings of matched parentheses, we introduce two judgments \(s \texttt{mparen}\) and \(s \texttt{lparen}\) with the following inference rules:

\[
\frac{\quad}{\epsilon \texttt{mparen}} \quad M\texttt{eps} \quad \frac{s \texttt{mparen} \quad (s) \texttt{mparen}}{\texttt{Mpar}} \quad \frac{s_1 \texttt{mparen} \quad s_2 \texttt{mparen}}{s_1 s_2 \texttt{mparen} \quad M\texttt{seq}}
\]

\[
\frac{\quad}{\epsilon \texttt{lparen}} \quad L\texttt{eps} \quad \frac{s_1 \texttt{lparen} \quad s_2 \texttt{lparen} \quad (s_1) s_2 \texttt{lparen}}{\texttt{Lseq}}
\]

Give a proof of Theorem 0.1. If you need a lemma to complete the proof, state the lemma, prove it, and use it in your proof of Theorem 0.1.

\textbf{Theorem 0.1.} If \(s \texttt{mparen}\), then \(s \texttt{lparen}\).

Please see the course notes for a proof.
λ-calculus [60 points]

Question 1. [10 points]
Write the abstract syntax of the untyped λ-calculus:

expression e ::= x | λx.e | e e

Question 2. [20 points] Show the reduction sequence under the call-by-name strategy. Underline the redex at each step.

\[(\lambda x. \lambda y. y \ x) \ ((\lambda x. \ x) \ (\lambda y. \ y)) \ (\lambda z. \ z)\]

\[\mapsto (\lambda y. \ y \ ((\lambda x. \ x) \ (\lambda y. \ y))) \ (\lambda z. \ z)\]

\[\mapsto (\lambda z. \ ((\lambda x. \ x) \ (\lambda y. \ y)))\]

\[\mapsto (\lambda x. \ x) \ (\lambda y. \ y)\]

\[\mapsto (\lambda y. \ y)\]

Question 3. [30 points] Complete the three reduction rules for the call-by-value strategy. You may use the substitution \([e'/x]e\) as given in the course notes.

\[
\begin{align*}
\frac{e_1 \mapsto e_1'}{e_1 e_2 \mapsto e_1' e_2} & \quad \text{Lam} \\
\frac{e_2 \mapsto e_2'}{(\lambda x. e) e_2 \mapsto (\lambda x. e) e_2'} & \quad \text{Arg} \\
\frac{(\lambda x. e) v \mapsto [v/x]e}{(\lambda x. e) v \mapsto [v/x]e} & \quad \text{App}
\end{align*}
\]